Section 8: Conic Sections, Parametric Equations, and Polar Coordinates

The following maps the videos in this section to the Texas Essential Knowledge and Skills for Mathematics TAC §111.42(c).

8.01 Conic Sections

- Precalculus (3)(F)

8.02 Ellipses

- Precalculus (3)(G)
- Precalculus (3)(H)

8.03 Hyperbolas

- Precalculus (3)(G)
- Precalculus (3)(I)

8.04 Parametric Equations

- Precalculus (3)(A)
- Precalculus (3)(B)
- Precalculus (3)(C)

8.05 Polar Coordinates and Equations

- Precalculus (3)(D)

8.06 Polar Graphs

- Precalculus (3)(E)

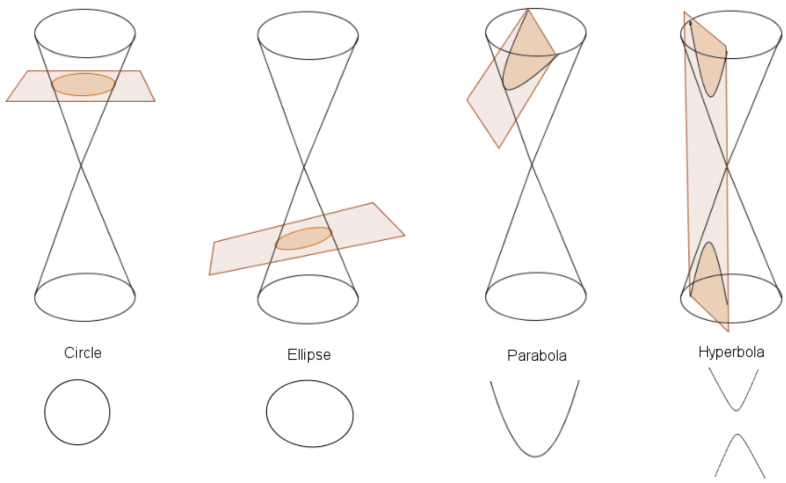
8.07 Special Polar Graphs

- Precalculus (3)(E)

Note: Unless stated otherwise, any sample data is fictitious and used solely for the purpose of instruction.

8.01

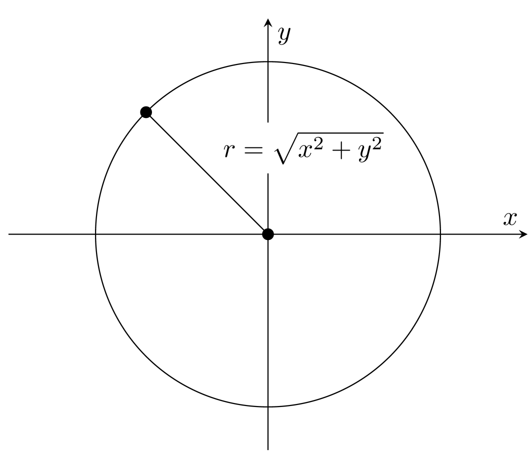
Conic Sections



Conic section – The graph obtained by the intersection of a plane and a double-napped cone

- The four main conic sections are the circle, parabola, ellipse, and hyperbola. These occur when the intersecting plane does **not** pass through the vertex of the cone.
- Aside from the geometric definition, conic sections can be defined algebraically in terms of a general second-degree equation: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
- One approach is to define each conic section as a ***locus*** (collection) of points that satisfy a particular geometric property.

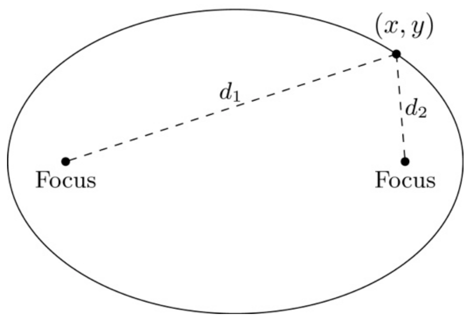
For example, a circle is defined as the set of all points (x, y) in a plane that are equidistant from a fixed point (h, k). That distance is known as the radius of the circle, and it leads to the standard form of a circle: $(x-h)^2 + (y-k)^2 = r^2$

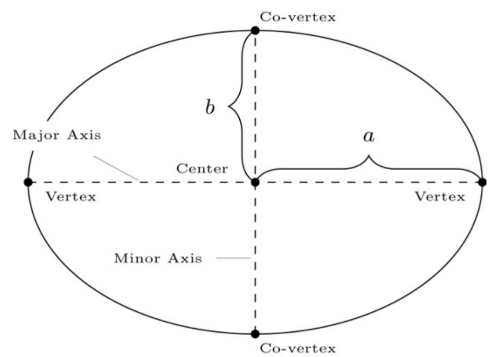


8.02

Ellipses

Ellipse – The set of all points (x, y) in a plane such that the sum of their distances from two *distinct* fixed points is constant





Focus – Each fixed point in the definition of an ellipse (plural: ***foci***)

Vertices – The points at which the line through the foci intersects the ellipse

Co-vertices –The points at which the line perpendicular to the line through the foci and passing through the center intersects the ellipse

Major axis – The longer axis of symmetry

- Formally, the major axis is the chord joining the vertices.
- The foci always lie on the major axis.

Minor axis – The shorter axis of symmetry

Formally, the minor axis is the chord perpendicular to the major axis that passes through the center.

Center – The midpoint of both the major axis and the minor axis

The table below describes the ***standard form*** of the equation of an ellipse with center (h,k).
The equations satisfy $a > b$ and $c^2 = a^2 - b^2$.

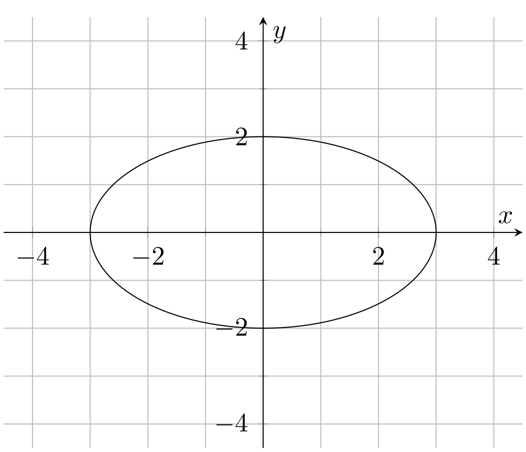
Ellipse	Major Axis	Length of Major Axis	Length of Minor Axis	Vertices	Co-vertices	Foci
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$	Horizontal	$2a$	$2b$	$(h \pm a, k)$	$(h, k \pm b)$	$(h \pm c, k)$
$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$	Vertical	$2a$	$2b$	$(h, k \pm a)$	$(h \pm b, k)$	$(h, k \pm c)$

STUDY EDGE TIP

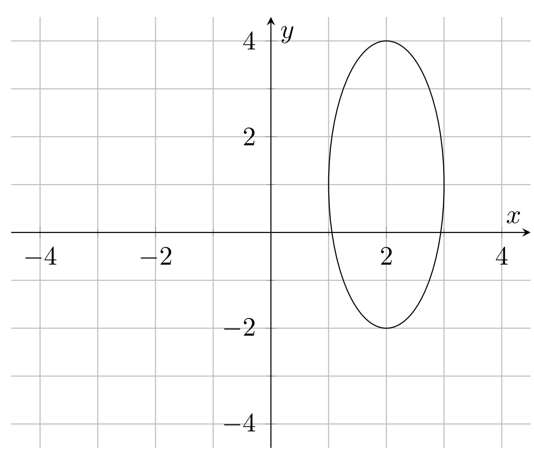
Foci lie on the major axis c units from the center, where $c^2 = a^2 - b^2$.
Vertices lie on the major axis a units from the center.

1. Find the standard form of the equation of each ellipse below:

i.



ii.



2. Consider an ellipse with the equation $\frac{(x-2)^2}{30}+\frac{(y+3)^2}{36}=1$.

   i. Find the center.

   ii. Find the foci.

   iii. Find the vertices.

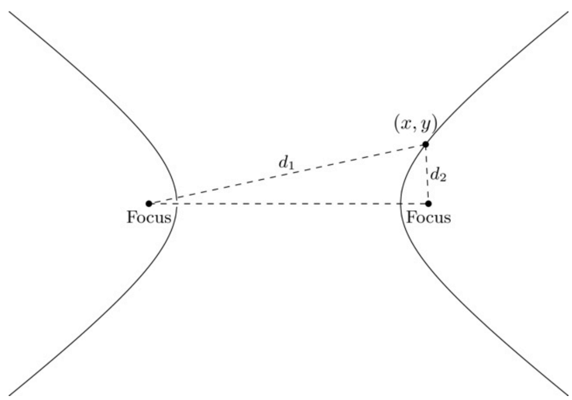
3. Find the standard form of the equation of the ellipse that has the center $(0, 2)$, foci at $(0, 5)$ and $(0, -1)$, and a vertex at $(0, 7)$.

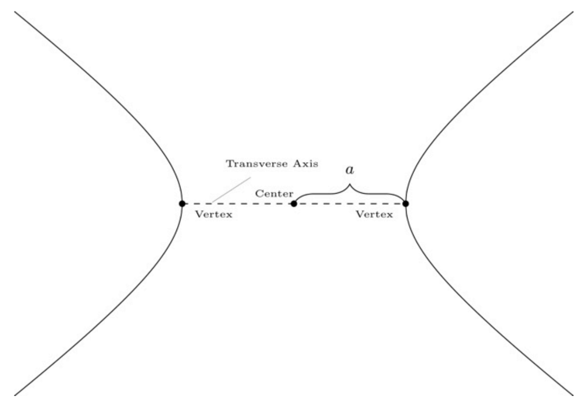
4. Determine whether $9x^2 + 4y^2 + 36x - 24y + 36 = 0$ is a circle or an ellipse. Then, find the center, radius, and foci of the conic (as applicable), and sketch its graph.

8.03

Hyperbolas

Hyperbola – The set of all points (x, y) in a plane such that the difference of the distances from two *distinct* fixed points is a positive constant





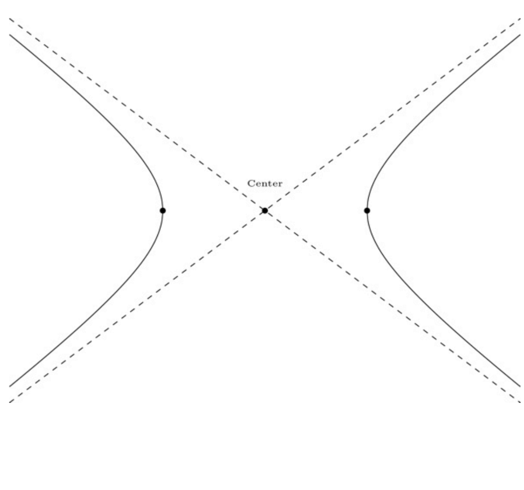
Focus – Each fixed point in the definition of a hyperbola (plural: ***foci***)

Vertices – The points at which the line through the foci intersects the hyperbola

Transverse axis – The line segment joining the vertices, passing through the center of the hyperbola

Center – The midpoint of the transverse axis

Every hyperbola has two asymptotes that pass through the center:



The following table describes the ***standard form*** of the equation of a hyperbola with center (h,k).

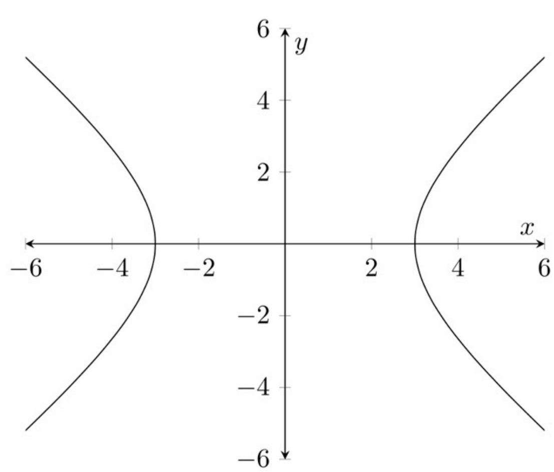
The equations satisfy $c^2 = a^2 + b^2$.

Hyperbola	Transverse Axis	Length of Transverse Axis	Vertices	Foci	Asymptotes
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	Horizontal	$2a$	$(h \pm a, k)$	$(h \pm c, k)$	$y = \pm\frac{b}{a}(x-h)+k$
$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$	Vertical	$2a$	$(h, k \pm a)$	$(h, k \pm c)$	$y = \pm\frac{a}{b}(x-h)+k$

STUDY EDGE TIP

Foci lie on the transverse axis c units from the center, where $c^2 = a^2 + b^2$. Vertices lie on the transverse axis a units from the center.

1. Find the standard form of the equation of the hyperbola given that $(3,0)$, $(-3,0)$ and $(4,2)$ are points on the graph.



2. Consider a hyperbola with the equation $\frac{x^2}{25}-\frac{y^2}{49}=1$.

   i. Find the center.

   ii. Find the foci.

   iii. Find the vertices.

3. Consider a hyperbola with the equation $\frac{(y+3)^2}{25}-\frac{(x+4)^2}{11}=1$.

   i. Find the center.

   ii. Find the foci.

   iii. Find the vertices.

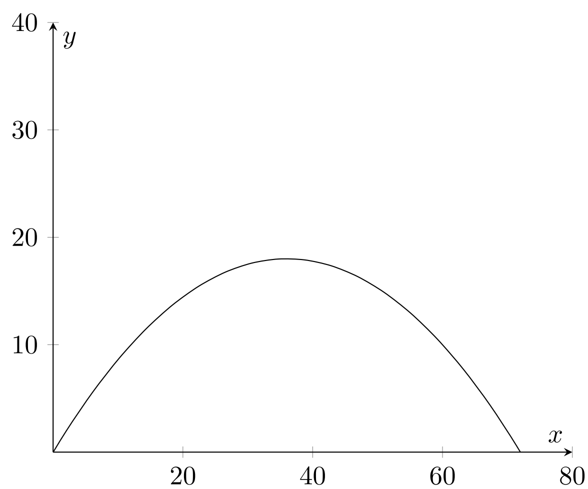
4. Find the standard form of the equation of the hyperbola that has the center $(2, 1)$, a focus at $(8, 1)$, and a vertex at $(-3, 1)$.

5. Find the equations of the asymptotes of the hyperbola $\frac{(x-4)^2}{25} - \frac{(y+2)^2}{16} = 1$.

8.04
Parametric Equations

Rectangular equation graph – Any graph with an equation that takes the form $y = f(x)$

A rectangular graph consists of a single equation involving two variables, x and y. As an example, consider the graph of $y = -\frac{x^2}{72} + x$, which describes the position of an object that has been propelled into the air at an angle of 45° at an initial velocity of 48 feet per second.



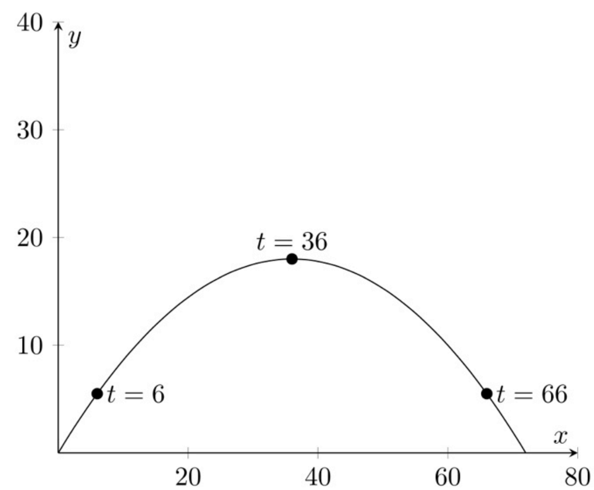
The (rectangular) equation $y = -\frac{x^2}{72} + x$ describes exactly *where* the object has been.

However, it leaves out important information, namely, *when* the object was at a given point (x, y) and which *direction* the object was travelling in. To determine *when* the object was at a certain point, we must introduce a third variable, t.

Parameter – The variable t used to define the parametric equations $x = g(t)$ and $y = h(t)$

Parametric equations – The equations $x = g(t)$ and $y = h(t)$, which are used to describe the position of an object at a point $(x(t), y(t))$ for any given parameter value t

- Instead of defining y as a function of x, we define both x and y as functions of the parameter t.
- The parameter t is often thought of as time; however, t may be any real value satisfying $-\infty < t < \infty$.
- Technically, any variable may be used in place of t as the parameter.



Plane curve – If $f(t)$ and $g(t)$ are continuous functions of t on an interval I, the set of ordered pairs $(f(t), g(t))$ is a plane curve C.

- Each value of t defines a point $(x, y) = (f(t), g(t))$ that can be plotted in the coordinate plane. As t varies over all values on the interval I, the points trace out a plane curve, sometimes called a ***parametric curve***.
- Each parametric curve traces out its graph in a particular direction, either forward or backward.
- The direction of motion follows the direction of increasing t. To find the direction of motion, pick at least three values of t (that *increase*) and determine their positions on the curve. Then, draw an arrow to show the direction of motion.

Eliminating the parameter – The process by which a parametric equation is converted into a rectangular equation

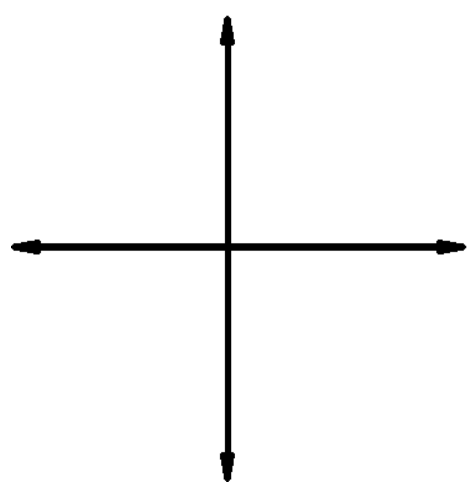
One way to eliminate the parameter is to, follow these steps:

1. Solve one of the parametric equations for the parameter.
2. Substitute the solved parameter from Step 1 into the other equation.

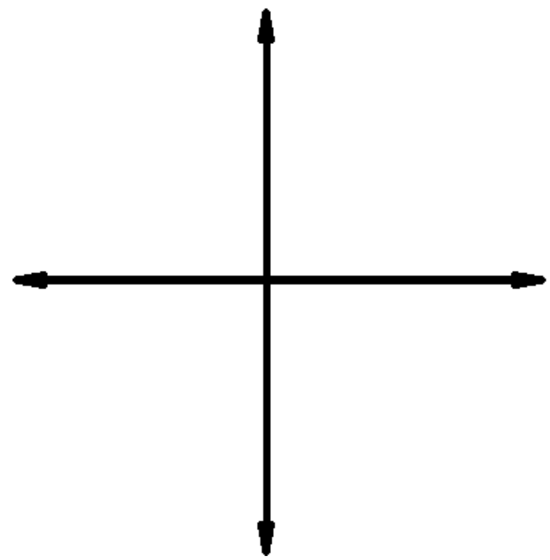
1. Consider a plane curve with the parametric equations $x(t) = 2t$ and $y(t) = t^2 - 1$ for $-2 < t < 2$.

   i. Eliminate the parameter and write the corresponding rectangular equation.

   ii. Graph the plane curve represented by the parametric equations and indicate its orientation. Adjust the domain of the rectangular equation if necessary.



2. A drone follows the trajectory $x(t) = \cos t$, $y(t) = \sin t$, $0 \le t \le 2\pi$. Eliminate the parameter and then graph the resulting rectangular curve.



3. How would the rectangular curve in the previous problem change if $x(t) = 2\cos t + 4$, $y(t) = 3\sin t - 1$, and $0 \le t \le 2\pi$?

Projectile motion – A projectile launched at a height of h feet above the ground, at an angle θ with the horizontal, with initial velocity v_0 feet per second, has a path defined by the parametric equations $x = (v_0 \cos\theta)t$ and $y = h + (v_0 \sin\theta)t - 16t^2$.

4. A disc is thrown from a point 10 feet above the ground. The disc leaves the thrower's hand at an angle of 30° with the horizontal and at an initial speed of 25 feet per second.

   i. Write a set of parametric equations that model the path of the disc.

   ii. Assuming the ground is level, find the distance the disc travels before it hits the ground.

   iii. Find the total time the disc is in the air.

   iv. Use a graphing utility to graph the path of the disc and approximate its maximum height.

8.05

Polar Coordinates and Equations

Rectangular coordinate system – Specifies a point (x, y) in a plane as its directed signed distance from the both the x-axis and the y-axis

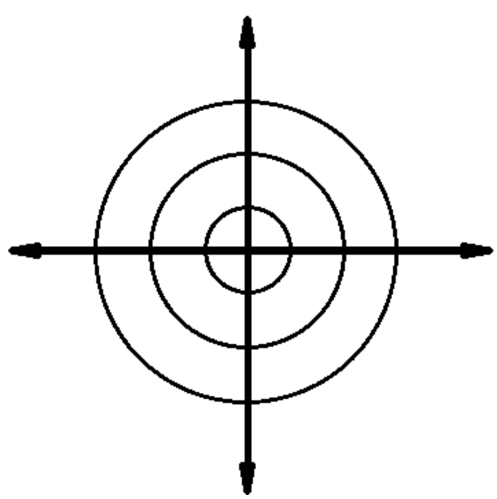
- This is the coordinate system you've been using already when graphing.
- The point (x, y) is represented by a pair of numerical values, with x representing the signed distance from the x-axis and y representing the signed distance from the y-axis.
- The rectangular coordinate system is also called the ***Cartesian coordinate system***.

1. Plot the point $(-3,2)$ in the rectangular coordinate system.

The ***polar coordinate system*** defines points with an angle and a single distance.

Polar coordinate (r,θ) – Specifies a point in a plane as its directed distance, r, from the origin and its directed angle, θ, made with the positive x-axis (polar axis)

For example, the Cartesian coordinate $\left(1,\sqrt{3}\right)$ is equivalent to the polar coordinate $\left(2,\frac{\pi}{3}\right)$.

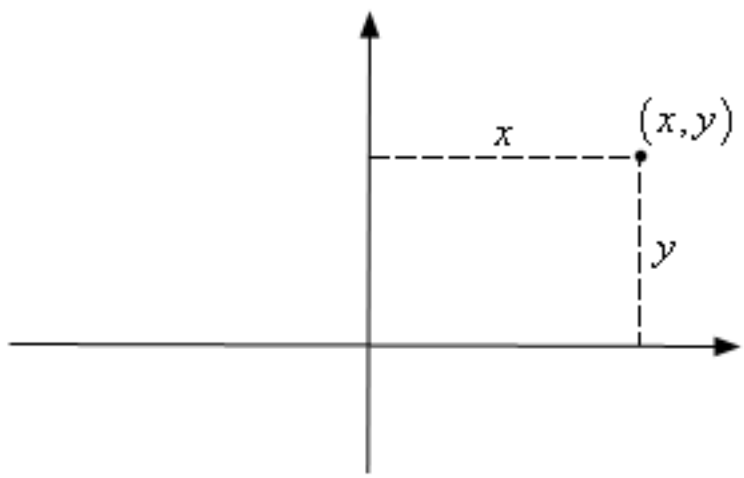


Pole – A fixed point that is used as the starting position for a polar coordinate

- The pole is equivalent to the origin $(0,0)$ in the rectangular coordinate system.
- The pole is defined as $(0,\theta)$ for any angle θ in the polar coordinate system.

Polar axis – The initial ray extending from the pole, corresponding to the positive x-axis

Polar coordinates use right triangle trigonometry to define coordinate transformations.



Conversion from Cartesian to Polar	Conversion from Polar to Cartesian
$r^2 = x^2 + y^2$ $\tan\theta = \frac{y}{x}$ if $x \neq 0$	$x = r\cos(\theta)$ $y = r\sin(\theta)$

- In rectangular coordinates, every point (x,y) has a unique representation.
- In polar coordinates, there are many representations of the same point because sine and cosine are periodic.
- In particular, $(r,\ \theta) = (r,\ \theta \pm 2\pi n)$ or $(r,\ \theta) = (-r,\ \theta \pm (2n+1)\pi)$, where n is any integer.
- Keep in mind that if r is negative, the distance goes in the opposite direction of θ.

2. Which of the following is a possible polar coordinate for the point P with rectangular coordinates $(0,-6)$?

A. $(r,\theta)=\left(6,\dfrac{\pi}{2}\right)$

B. $(r,\theta)=\left(-6,-\dfrac{3\pi}{2}\right)$

C. $(r,\theta)=\left(-6,\dfrac{7\pi}{2}\right)$

D. $(r,\theta)=\left(-6,-\dfrac{\pi}{2}\right)$

E. $(r,\theta)=\left(6,-\dfrac{7\pi}{2}\right)$

3. Convert the polar coordinates $(r,\theta)=\left(2,\dfrac{\pi}{3}\right)$ to Cartesian coordinates.

STUDY EDGE TIP

- When converting rectangular to polar, if a point is in the second or third quadrant, then $\theta = \pi + \tan^{-1}\left(\frac{y}{x}\right)$ if we want $\theta, r > 0$.
- This is equivalent to using $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ and negating r.

4. Convert the Cartesian coordinates $(x,y)=\left(-\frac{7}{\sqrt{3}},7\right)$ to polar coordinates with positive r and $0 \le \theta < 2\pi$.

$\tan\theta$	θ
$-\sqrt{3}$	$-\pi/3$
$-1/\sqrt{3}$	$-\pi/6$
$1/\sqrt{3}$	$\pi/6$
$\sqrt{3}$	$\pi/3$

For Problems 5–7, convert the rectangular equation to a polar equation.

5. $y = 4$

6. $x^2 + y^2 = 9$

7. $x - y = 5$

For Problems 8–10, convert the polar equation to a rectangular equation.

8. $r = 5$

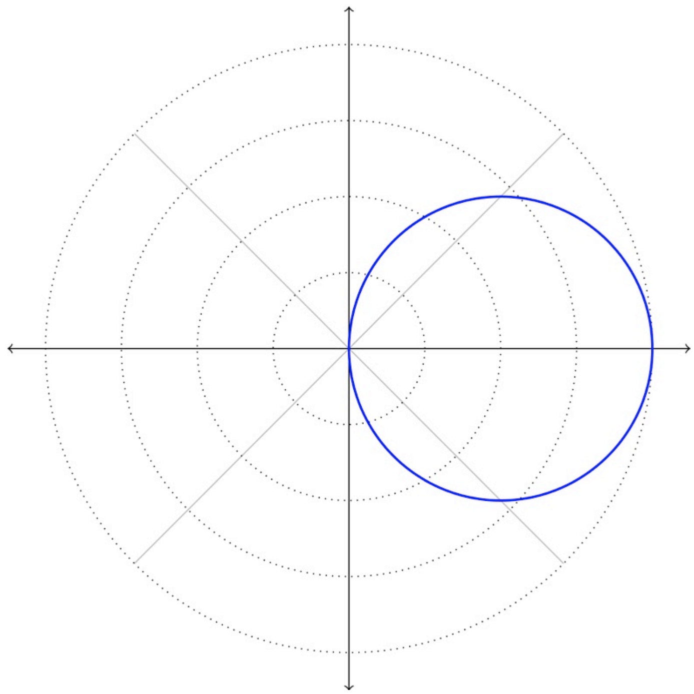
9. $r = \csc\theta$

10. $r = 4\sin\theta + 9\cos\theta$

8.06

Polar Graphs

Consider the graph below of the polar equation $r = 4\cos\theta$.

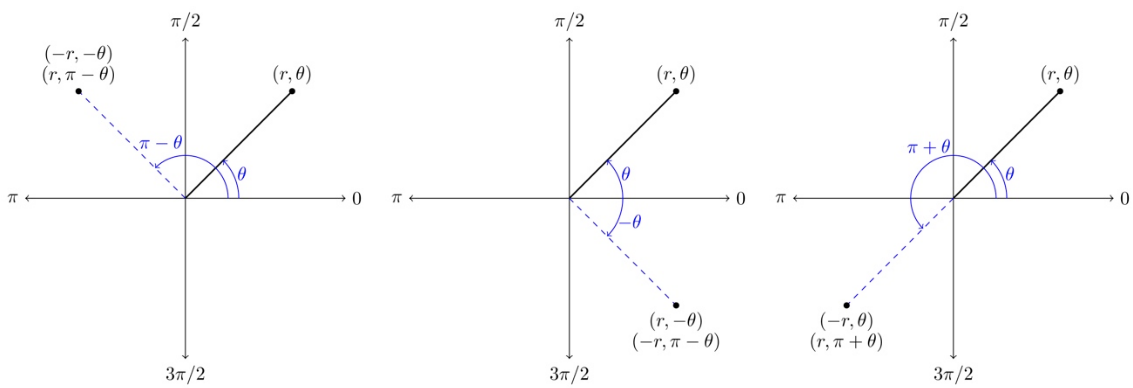


θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r					

Notice that this graph has symmetry over the polar axis. In fact, we can categorize when a polar graph will have certain symmetries and use this to our advantage when graphing.

The graph of a polar equation is symmetric with respect to the following if the given substitution yields an equivalent equation.

Type of Symmetry	**Replace (r,θ) by**
The line $\theta=\dfrac{\pi}{2}$	$(-r,-\theta)$
The polar axis	$(r,-\theta)$
The pole	$(-r,\theta)$



Quick Tests for Symmetry in Polar Graphs

1. Equations involving only sine are symmetric with the line $\theta=\dfrac{\pi}{2}$.
2. Equations involving only cosine are symmetric with the polar axis.

It is also helpful to know the θ values for which $|r|$ is maximized and for which $r=0$.

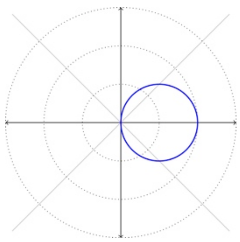
1. Suppose $r = 4\cos\theta$. Find the values θ for which $|r|$ is maximized and for which $r = 0$ on the interval $[0, 2\pi)$.

2. Sketch the graph of $r = 1 - 2\cos\theta$ using symmetry, zeros, maximum r-values, and any other additional points.

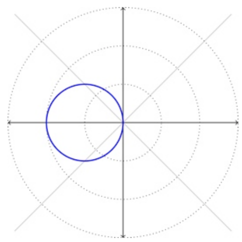
<u>8.07</u>

<u>Special Polar Graphs</u>

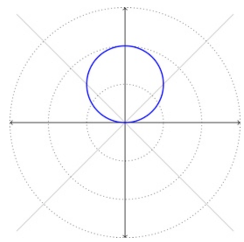
Circle



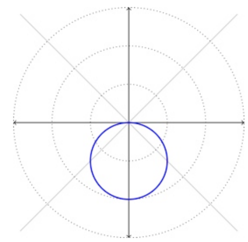
$r = a\cos(\theta)$, $a > 0$



$r = a\cos(\theta)$, $a < 0$



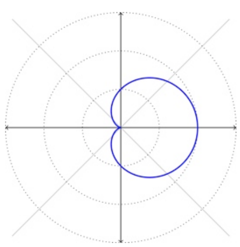
$r = a\sin(\theta)$, $a > 0$



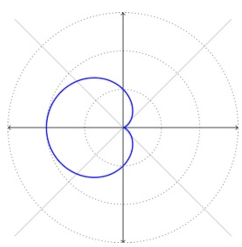
$r = a\sin(\theta)$, $a < 0$

These circles have radii of length $\frac{|a|}{2}$. The center of $r = a\cos(\theta)$ is $\left(\frac{a}{2}, 0\right)$. The center of $r = a\sin(\theta)$ is $\left(0, \frac{a}{2}\right)$.

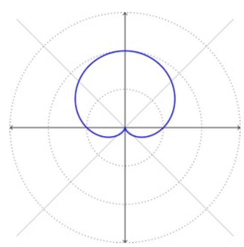
Cardioid



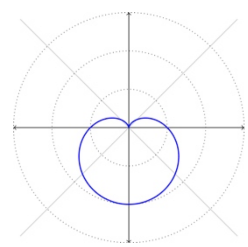
$r = a + b\cos(\theta)$



$r = a - b\cos(\theta)$



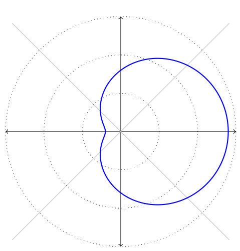
$r = a + b\sin(\theta)$



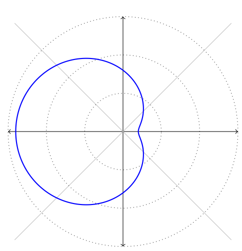
$r = a - b\sin(\theta)$

For these graphs $a > 0$, $b > 0$, and $\frac{a}{b} = 1$.

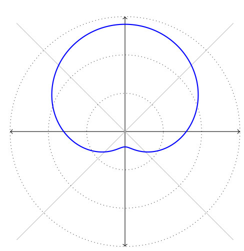
Limaçon with one loop



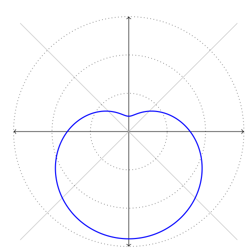
$r = a + b\cos(\theta)$



$r = a - b\cos(\theta)$



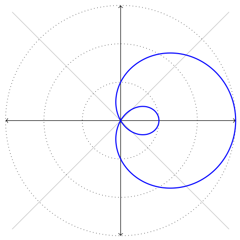
$r = a + b\sin(\theta)$



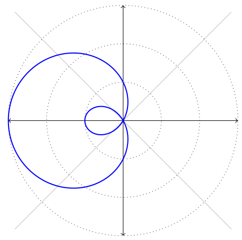
$r = a - b\sin(\theta)$

For these graphs $a > 0$, $b > 0$, and $1 < \frac{a}{b} < 2$.

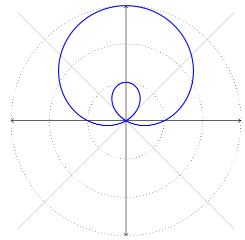
Limaçon with an inner loop



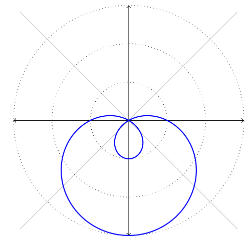
$r = a + b\cos(\theta)$



$r = a - b\cos(\theta)$



$r = a + b\sin(\theta)$



$r = a - b\sin(\theta)$

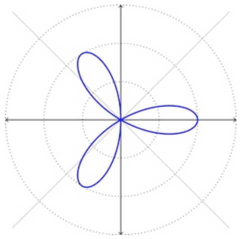
For these graphs $a > 0$, $b > 0$, and $a < b$.

For the graphs of $r = a \pm b\cos(\theta)$ or $r = a \pm b\sin(\theta)$, set $r = 0$ and solve on the interval $\left[0, 2\pi\right)$. There are three possibilities:

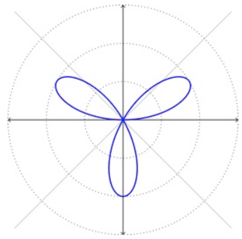
1. Setting $r = 0$ yields one of $\theta = 0,\ \frac{\pi}{2},\ \pi, \text{ or } \frac{3\pi}{2}$ as a solution. This graph is a cardioid.
2. Setting $r = 0$ yields no solutions. This graph is a limaçon with one loop.
3. Setting $r = 0$ yields two distinct solutions. This graph is a limaçon with an inner loop.

1. Determine the type of polar graph described by $r = 3 - 2\cos\theta$.

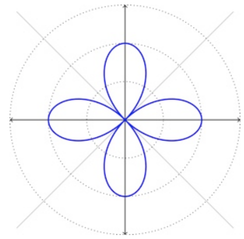
Rose Curves



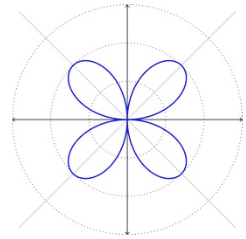
$r = a\cos(3\theta)$



$r = a\sin(3\theta)$



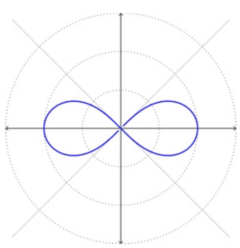
$r = a\cos(2\theta)$



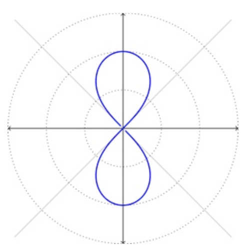
$r = a\sin(2\theta)$

In general, these equations take on the form $r = a\cos(n\theta)$ and $r = a\sin(n\theta)$. If n is even, there are $2n$ petals, and if n is odd, there are n petals.

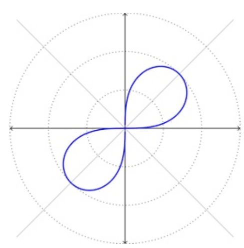
Lemniscates



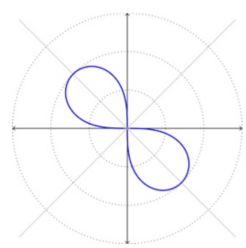
$r^2 = a\cos(2\theta)$



$r^2 = -a\cos(2\theta)$



$r^2 = a\sin(2\theta)$



$r^2 = -a\sin(2\theta)$

2. Match the given polar equations to the graphs.

A. $r = 1 - 2\cos\theta$
B. $r = 2\sin(4\theta)$
C. $r = 3 + 2\sin\theta$
D. $r = 2$
E. $r = 2 - 3\sin\theta$
F. $r^2 = \sin(2\theta)$