Section 6: Trigonometric Identities and Applications

The following maps the videos in this section to the Texas Essential Knowledge and Skills for Mathematics TAC §111.42(c).

6.01 Trigonometric Identities
- Precalculus (5)(M)
- Precalculus (5)(N)

6.02 Solving Trigonometric Equations
- Precalculus (5)(M)
- Precalculus (5)(N)

6.03 Sum and Difference Formulas of Trigonometric Functions
- Precalculus (5)(M)
- Precalculus (5)(N)

6.04 Double Angle and Half Angle Formulas of Trigonometric Functions

6.05 Law of Sines and Cosines
- Precalculus (4)(G)
- Precalculus (4)(H)

6.06 Trigonometric Word Problems
- Precalculus (4)(E)
- Precalculus (4)(F)
- Precalculus (5)(M)
- Precalculus (5)(N)

6.07 Vectors
- Precalculus (4)(I)
- Precalculus (4)(J)
- Precalculus (4)(K)

6.01

Trigonometric Identities

Reciprocal Identities

$\sin\theta = \frac{1}{\csc\theta}$ $\cos\theta = \frac{1}{\sec\theta}$ $\tan\theta = \frac{1}{\cot\theta}$

Quotient Identities

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Cofunction Identities

$\sin\theta = \cos(90-\theta)$ $\sec\theta = \csc(90-\theta)$ $\tan\theta = \cot(90-\theta)$

$\cos\theta = \sin(90-\theta)$ $\csc\theta = \sec(90-\theta)$ $\cot\theta = \tan(90-\theta)$

Negative Angle Identities (Even/Odd Functions)

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

1. Given $\sec\left(\frac{\pi}{2}-\theta\right) = 3$ and $\cos\theta > 0$, find $\cot\theta$.

STUDY EDGE TIP

Below are some guidelines for matching or verification problems:

Step 1: Start with any identities.

Step 2: Use the methods of factoring, common denominators, separating numerators, and conjugates.

Step 3: If nothing is working, try changing everything to $\sin\theta$ and $\cos\theta$.

2. Verify $\frac{1-\cos x}{1+\cos x} = 2\csc^2 x - 2\csc x \cot x - 1$.

3. Verify $\frac{\csc x+1}{\csc x-1} = \frac{1+\sin x}{1-\sin x}$.

6.02

Solving Trigonometric Equations

For these equations, more than one solution may exist, or there may be no solution.

1. Solve for all angles A: $2\cos A - 1 = 0$

STUDY EDGE TIP

Below are steps to use when asked to solve for all angles:

Step 1: Solve for angles within the specified interval.

Step 2: If the angles are separated by π radians, take the smallest of the two angles and add πn, where n is an integer.

Step 3: If the angles are not separated by π radians, take each angle and add $2\pi n$, where n is an integer.

2. Solve $\cos x \cot x = \cos x$ for all angles.

3. Solve $\cos^2 A - 4\cos A + 3 = 0$ in the interval $[0, 2\pi)$.

6.03

Sum and Difference Identities

$\sin(u+v) = \sin u \cos v + \cos u \sin v$

$\sin(u-v) = \sin u \cos v - \cos u \sin v$

$\cos(u+v) = \cos u \cos v - \sin u \sin v$

$\cos(u-v) = \cos u \cos v + \sin u \sin v$

$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$

$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$

1. Find $\cos 75°$ using sum and difference identities.

2. Find $\tan(A+B)$ if $\sin A = \frac{4}{5}$, A is in Quadrant II, $\sin B = -\frac{12}{13}$, and B is in Quadrant IV.

6.04
Double-Angle and Half-Angle Identities

Double-Angle Formulas

$$\sin 2u = 2\sin u \cos u$$
$$\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$$
$$\tan 2u = \frac{2\tan u}{1 - \tan^2 u}$$

1. If $\sin B = -\frac{1}{3}$ and $\tan B < 0$, find $\sin 2B$.

2. Solve $\sin 2A = \cos A$ on the interval $[0,2\pi)$.

Half-Angle Formulas

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$

$$\tan\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{1+cosu}} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

3. Find the exact value of $\sin 105°$ using half-angle formulas.

4. Find the exact value of $\cos\frac{x}{2}$ if $\cos x = \frac{1}{5}$ and x is in the fourth quadrant.

6.05

Law of Sines and Cosines

The law of sines and cosines is used for ***oblique triangles***, which are triangles that do not have right angles.

Given angles A, B, and C of an oblique triangle with opposite sides a, b, and c, we have

Law of sines – $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Law of cosines – $a^2 = b^2 + c^2 - 2bc\cos A$ $\quad b^2 = a^2 + c^2 - 2ac\cos B$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

STUDY EDGE TIP

Here are some strategies to consider when deciding to use the law of sines or the law of cosines:

- Law of sines: Use when you are given two angles and one side.
- Law of cosines: Use when you are given all three sides *or* two sides and one angle.

1. Given an oblique triangle where $a = 4$, $b = 7$, and $c = 6$, find angle C.

2. Given an oblique triangle where $b = 4$, $c = 7$, and angle $C = 50$, find $\sin B$.

3. Find side c of an oblique triangle with side $a = 4$, angle $A = 50°$, and angle $B = 60°$.

We have two equations for the area of an oblique triangle:

Heron's formula for area – Given sides a, b, c and the semi-perimeter s, we have

$s = \frac{1}{2}(a + b + c)$ and the area $A = \sqrt{s(s-a)(s-b)(s-c)}$

Area of an oblique triangle – Given two sides and the angle between them, we have

$A = \frac{1}{2}ab\sin\gamma$ $A = \frac{1}{2}ac\sin\beta$ $A = \frac{1}{2}bc\sin\alpha$

4. Find the area of an oblique triangle with the properties $a = 4$, $b = 7$, and $C = 75°$.

5. Find the area of a triangle whose sides have lengths of 3 ft, 5 ft, and 6 ft.

6.06

Trigonometric Word Problems

STUDY EDGE TIP

Here are two tips when setting up word problems involving trigonometry:

- Angle of elevation and angle of depression are with respect to the x-axis.
- Draw a picture!

1. Suppose you lean an 8-foot ladder against a wall at a 60-degree angle of elevation. How high is the top of the ladder from the base of the wall?

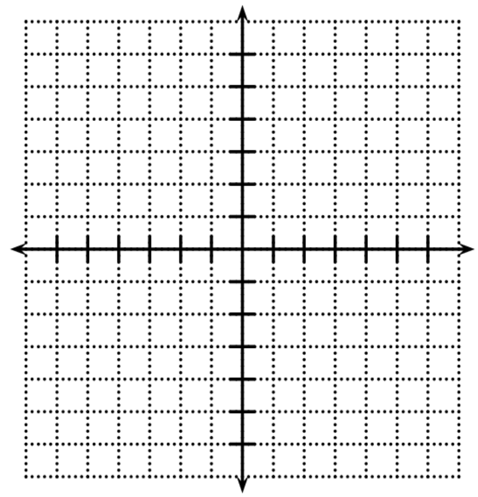
2. Imagine that a man is floating in the ocean, waiting to be rescued. He spots two helicopters on exactly opposite sides of him, one at a 30-degree angle of elevation and the other at a 60-degree angle of elevation. If the helicopters are both flying at an altitude of 100 meters, what is the distance between the helicopters?

3. A plane lifts off at an angle of elevation of 20 degrees at a rate of 300 feet per second. How many minutes pass before it reaches an altitude of 6,000 feet?

6.07

Vectors

A ***vector*** is a directed line segment in a plane.

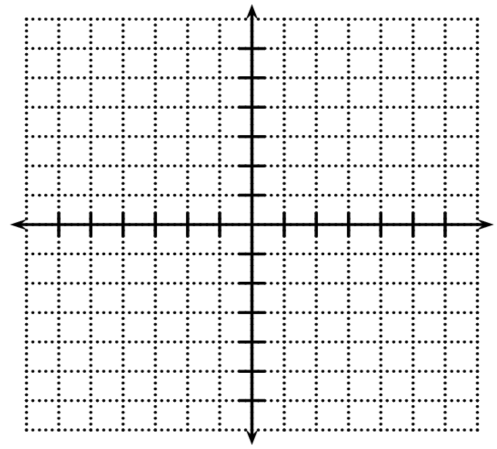


- A vector is a ***directed line segment*** and has an initial point and a terminal point.
- Vectors have a magnitude (length) and a direction.
  - Magnitude can be calculated using the distance formula.
  - Direction can be calculated using trigonometry.
- Vectors can be written as $v = \overrightarrow{PQ}$ where P and Q are two points on a plane.
- Vectors can also be written in ***component form*** as $v = \langle v_1, v_2 \rangle$.

1. Find the component form and magnitude of a vector with initial point (2,−5) and terminal point (−1,−9).

2. A ship sails from port and travels on a bearing of 30 degrees north of east at a speed of 20 nautical miles per hour. After three hours, how far east has the ship sailed?

Vector addition – Given two vectors $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$, then $u + v = \langle u_1 + v_1, u_2 + v_2 \rangle$.



Scalar multiplication – Given a vector $v = \langle v_1, v_2 \rangle$ and a constant c, then $cv = \langle cv_1, cv_2 \rangle$.

3. Given $u = \langle 5,8 \rangle$ and $v = \langle -3,7 \rangle$, find $u + v$, $3u$, and $3u - 4v$ in component form.

4. A cyclist started from her home and traveled four miles west, then three miles northeast, and finally two miles south. How far is the cyclist from her home?