Section 5: Introduction to Trigonometry and Graphs

The following maps the videos in this section to the Texas Essential Knowledge and Skills for Mathematics TAC §111.42(c).

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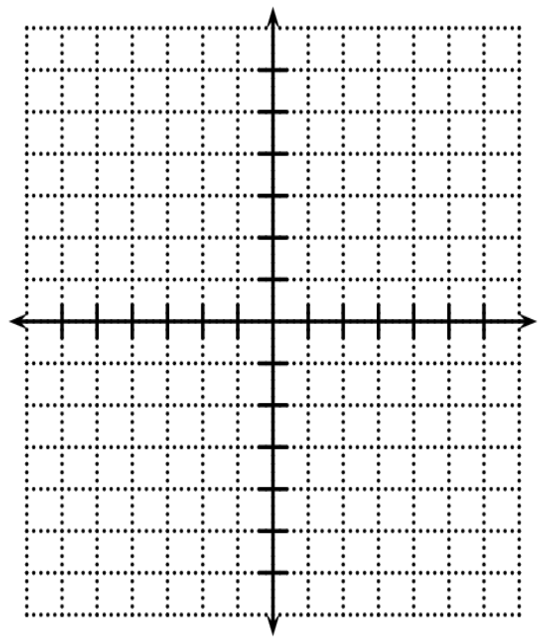
5.01
Radian and Degree Measurements

Trigonometry is the measure of triangles.

Angle – Two rays with a common endpoint

- An angle is usually denoted with the Greek letter θ, pronounced "theta."
- One of the rays is the ***initial side*** and typically starts on the positive x-axis at 0° (also known as ***standard position***).
- The other ray is known as the ***terminal side***.
- You will encounter many types of angles, including the following:
    - ***Positive angles*** – Rotate counterclockwise
    - ***Negative angles*** – Rotate clockwise
    - ***Quadrantal angles*** – The x- and y-axis angles
    - ***Coterminal angles*** – Two angles with the same initial and terminal sides
    - ***Complementary angles*** – Two acute angles whose sum is 90°
    - ***Supplementary angles*** – Two positive angles whose sum is 180°



The measurement of an angle is based on the amount of rotation from the initial side to the terminal side. Two common units of measure used for angles are ***degrees*** and ***radians***.

How many degrees are in a circle?

What is the equation for the circumference of a circle?

Using these two answers we can find a conversion between degrees and radians:

1. Convert 150° to radians.

2. Find the quadrant in which an angle measuring 420° is located.

3. Find the quadrant in which an angle measuring 3 radians is located.

4. Find the smallest positive coterminal angle for the angle $-\frac{17\pi}{6}$.

5. Which pair of angles is complementary?
    i. 10° and 80°
    ii. −10° and 100°

We use radians when measuring arc length and the area of a sector of a circle.

Arc length – Defined as $s = r\theta$

Area of a sector – Defined as $A = \frac{1}{2}r^2\theta$

- Both equations are derived from the circumference and area of a circle.
- In both equations, the ***central angle***, θ, is measured in radians.

6. A circle with a radius of 3 inches has an arc length of 6 inches. Find the central angle in radians.

7. A circle has a diameter of 12 feet and a central angle of 40°. What is the area of the sector?

5.02

Angular and Linear Velocity

Consider an object moving in a circular path at a constant speed. The distance, s, it travels along the circle in a given period of time, t, is the ***linear speed***, v. The angle (in radians) in which the same object rotates in a given period of time is the ***angular speed***, ω (pronounced "omega").

The equations associated with linear speed and angular speed are $v = \frac{s}{t}$ and $\omega = \frac{\theta}{t}$, respectively.

STUDY EDGE TIP

Aside from the equations for angular and linear speed, another way to convert between these speeds is to use unit conversions.

1. Find the angular speed, in revolutions per minute (rpm), of a top with a radius of 2 centimeters that is spinning 20 cm/min.

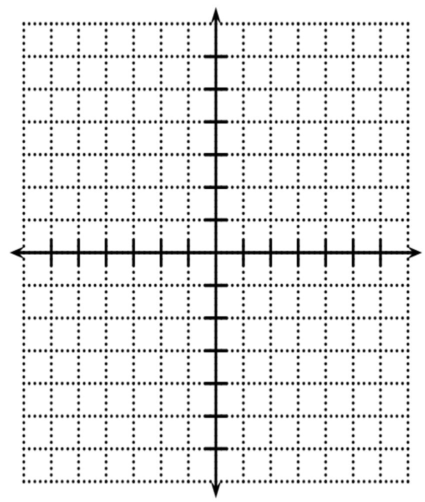
2. A merry-go-round with a diameter of 6 feet spins at an angle of 7 radians in 2 seconds. Find the angular speed in revolutions per minute.

5.03

Trigonometric Ratios

One application of trigonometry is to relate the ratio of two sides of a right triangle. This produces six combinations: ***sine, cosine, tangent, secant, cosecant,*** and ***cotangent***.

Given an acute angle A from standard position, we define the six trigonometric ratios as follows:



$\sin A = \frac{y}{r}$ $\quad \cos A = \frac{x}{r}$ $\quad \tan A = \frac{y}{x}$

$\csc A = \frac{r}{y}$ $\quad \sec A = \frac{r}{x}$ $\quad \cot A = \frac{x}{y}$

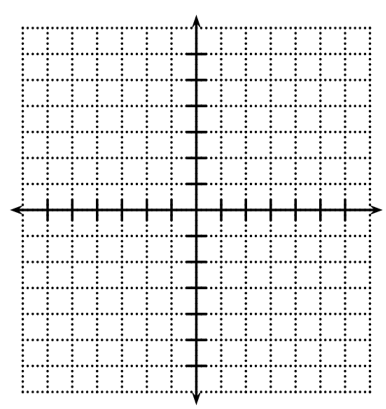
A more general rule for the trigonometric ratios relates them to the opposite side, adjacent side, and hypotenuse of a right triangle:

$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$ $\quad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$ $\quad \tan A = \frac{\text{opposite}}{\text{adjacent}}$

$\csc A = \frac{\text{hypotenuse}}{\text{opposite}}$ $\quad \sec A = \frac{\text{hypotenuse}}{\text{adjacent}}$ $\quad \cot A = \frac{\text{adjacent}}{\text{opposite}}$

And since we are discussing right triangles, recall the Pythagorean theorem $x^2 + y^2 = r^2$.

The quadrant the angle is in will determine whether the trigonometric ratio is positive or negative.



One way to remember the sign of various trigonometric ratios is by using the phrase ***All Students Take Calculus***.

STUDY EDGE TIP

When given one trigonometric ratio and asked to find another, the steps to solve are as follows:

1) Calculate x, y, and r.
2) Calculate the other trigonometric ratio.
3) Double check whether your final answer is positive or negative based on the quadrant it is in.

1. Find $\cos A$ if $\sin A = \frac{3}{4}$ and A is in quadrant II.

2. Find $\sin B$ if $\tan B = -\frac{5}{4}$ and $\sec B > 0$.

5.04

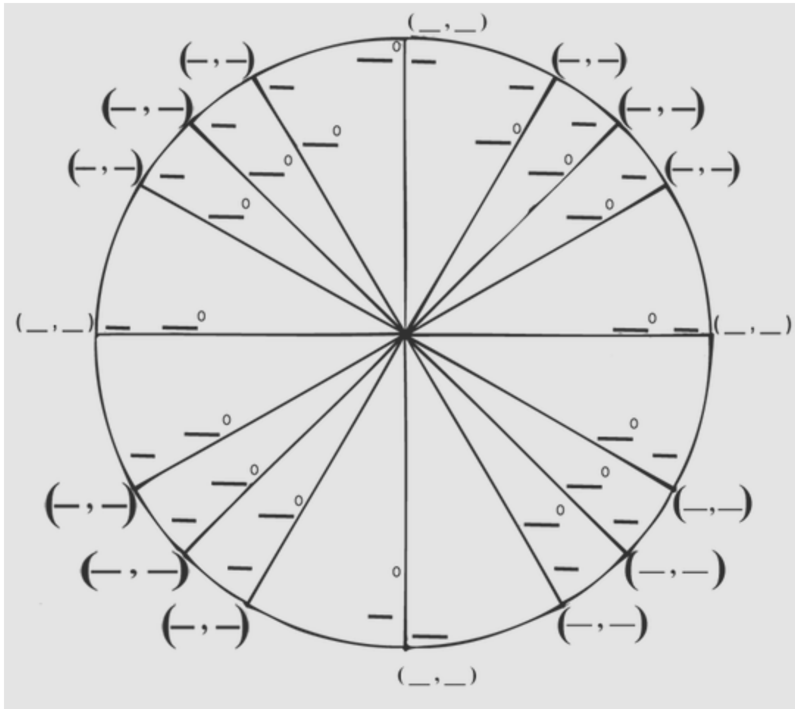
Trigonometric Angles and the Unit Circle

Let's start by exploring some common triangles from geometry.

Isosceles right triangles have two congruent sides with corresponding angles of 45°. We can find the sine and cosine of 45° as follows:

Equilateral triangles have three congruent sides with three congruent angles of 60°. Bisecting an equilateral triangle will give us two triangles with angles of 30°, 60°, and 90°. We can then find sine and cosine of 30° and 60° as follows:

One approach to remembering these angles is the ***unit circle***.



Another approach is by organizing the angles in a chart:

Degrees	$\mathbf{0^o}$	$\mathbf{30^o}$	$\mathbf{45^o}$	$\mathbf{60^o}$	$\mathbf{90^o}$
Radians	$\mathbf{0}$	$\boldsymbol{\frac{\pi}{6}}$	$\boldsymbol{\frac{\pi}{4}}$	$\boldsymbol{\frac{\pi}{3}}$	$\boldsymbol{\frac{\pi}{2}}$
$\sin\theta$					
$\cos\theta$					

Now, to find the other four trigonometric ratios, we must consider the identities of the remaining trigonometric functions in relation to the sine and cosine:

$\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Remember: Use **A**ll **S**tudents **T**ake **C**alculus to determine whether the trigonometric ratio is positive or negative, and use the reference angle to determine which quadrant you are in.

Reference angle – The acute angle formed between the terminal side and the horizontal axis

1. Evaluate $\cos 225°$.

2. Evaluate $\csc(-120°)$.

3. Evaluate $\sin \frac{5\pi}{3}$.

4. Evaluate $\sec -\frac{5\pi}{4}$.

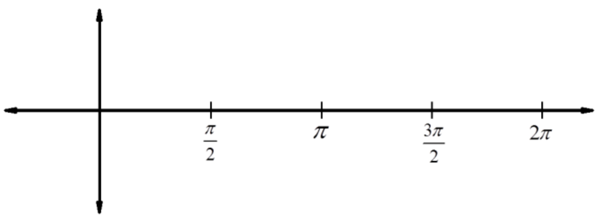
5. Evaluate $\cot \frac{19\pi}{6}$.

6. Suppose an 8-foot ladder leaning against a wall makes an angle of 30° with the wall. How far is the base of the ladder from the wall?

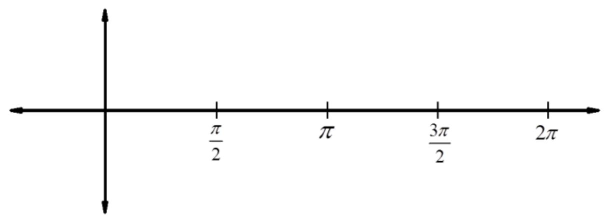
5.05

Graphs of Sine and Cosine

$y = \sin x$



$y = \cos x$



Key points – Five points on one period of a trigonometric function graph

Overall, a sine and cosine function will have the forms $y = d + a\sin(bx - c)$ and $y = d + a\cos(bx - c)$.

- ***Amplitude*** – The height of the wave, represented by a
  - The amplitude |a| is half the distance from the maximum to minimum y-value.
  - From the center, you would go up or down |a| units to find the crest or trough, respectively.
- ***Period*** – One cycle of the wave, equal to $\frac{2\pi}{b}$
  - $\sin(x + 2\pi) = \sin x$
  - $\cos(x + 2\pi) = \cos x$
- ***Vertical translation*** – Shift up (if positive) or down (if negative), represented by d
- ***Phase shift*** – Shift left (+) or right (−), equal to $\frac{c}{b}$
  - The phase shift is the starting location for one period of a trigonometric function.
  - To find the phase shift, set the inside equal to zero and solve for x.

Graphing Trigonometric Functions Using the Box Method

To graph a trigonometric function, we are going to employ the "box method":

A. Figure out the amplitude, period, and shifts.

   Step 1: Start at (0,0). Move up or down the necessary vertical translation and draw a horizontal dotted line. This is the middle of the box.

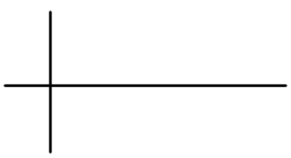
   Step 2: From the horizontal line you just drew, move up and down the given amplitude and draw horizontal dotted lines at these points. These are the top and bottom of the box.

   Step 3: From the point (0,0), go left or right the necessary phase shift and draw a vertical line. This is the start of the box.

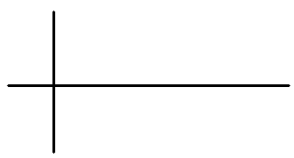
   Step 4: From the vertical line you just drew, go over one period and draw another vertical line. This is the end of the box. (In other words, you are adding the location of the phase shift plus the period.)

B. Draw the desired wave inside the box you have drawn.

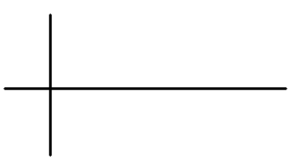
$y = \sin x$



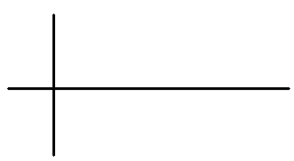
$y = -\sin x$



$y = \cos x$



$y = -\cos x$



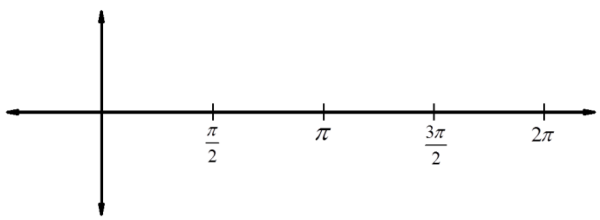
1. Find the five key points of $y = 3 - 2\sin x$ and graph one period of the equation.

2. The pistons in a car oscillate up and down like a sine wave. If a piston moves 0.42 inches from the bottom to the top of its oscillation in 0.3 seconds, determine the equation for the motion of the piston's height (in inches) in terms of time (in seconds). Assume that at $t = 0$ the piston starts in the middle ($y = 0$).

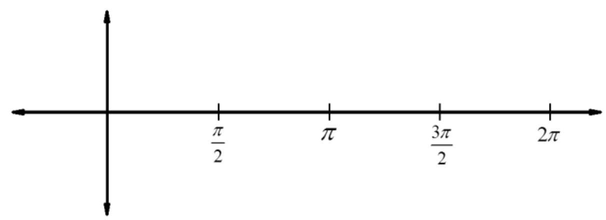
5.06

Graphs of Secant and Cosecant

$y = \sec x = \frac{1}{cos x}$



$y = \csc x = \frac{1}{\sin x}$



The general equations $y = d + a\ \sec(bx - c)$ and $y = d + a\ \csc(bx - c)$ have two new characteristics in addition to the characteristics already described for sine and cosine functions. These new characteristics are vertical asymptotes and domain restrictions.

Vertical asymptotes – Vertical dashed lines that mark the limits of a function graph, so that a curve may approach the asymptotes but never intersect them

- Vertical asymptotes of $y = csc\, x$ are located at $x = \pi n, n = 0, \pm 1, \pm 2, \ldots$
- Vertical asymptotes of $y = \sec x$ are located *at* $x = \frac{\pi}{2} + \pi n, n = 0, \pm 1, \pm 2, \ldots$

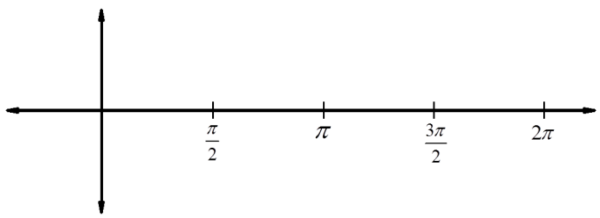
Domain restrictions – Inputs or points that are excluded when evaluating a function

1. Find the five key points of $y = 2 + \frac{1}{2}\cos\,(2x - \pi)$ and graph one period of the equation. Use this graph to then graph one period of $y = 2 + \frac{1}{2}\sec\,(2x - \pi)$.

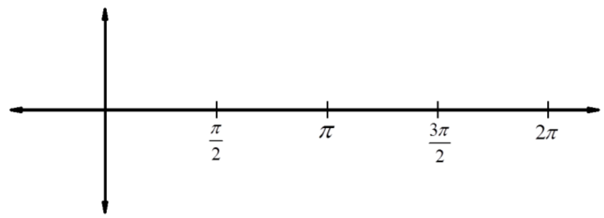
5.07

Graphs of Tangent and Cotangent

$y = \tan x$



$y = \cot x$



The general equations $y = d + a\tan(bx - c)$ and $y = d + a\cot(bx - c)$ have the following characteristics:

- The period for tangent and cotangent is $\frac{\pi}{b}$.
  - $\tan(x + \pi) = \tan x$
  - $\cot(x + \pi) = \cot x$
- Tangent and cotangent functions have vertical asymptotes and domain restrictions as follows:
  - Vertical asymptotes of $y = \cot x$ are located at $x = \pi n, n = 0, \pm 1, \pm 2, \dots$
  - Vertical asymptotes of $y = \tan x$ are located at $x = \frac{\pi}{2} + \pi n, n = 0, \pm 1, \pm 2, \dots$

STUDY EDGE TIP

To find two asymptotes of a trigonometric function graph, set the inside equal to two consecutive asymptotes of the original graph.

1. Graph one period of $y = 3\tan(3x - \pi)$.

2. Graph two periods of $y = 3\cot\left(\pi x - \frac{\pi}{4}\right)$.

3. Find all the vertical asymptotes of $y = -2\tan\left(x - \frac{\pi}{4}\right)$.

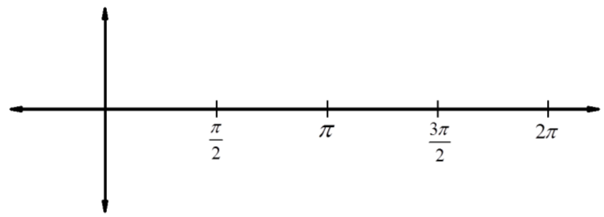
4. Find all the x-intercepts of $y = 2\cot(4x + \pi)$.

5.08

Inverse Trigonometry

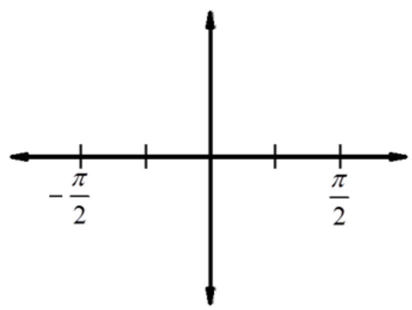
Recall that an inverse function must pass the horizontal line test:

If we graph $y = \sin x$, we have the following:



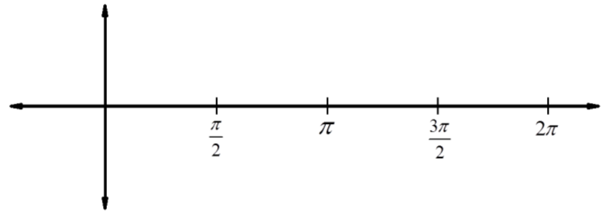
Sometimes we need to restrict the domain of $f(x)$ to obtain an inverse.

For $y = \sin x$, we can restrict it as follows:



And we can graph the inverse $y = \arcsin x$, also written as $y = \sin^{-1} x$:

For $y = \cos x$, we can restrict it as follows:



And we can graph the inverse $y = \arccos x$, also written as $y = \cos^{-1} x$:

Summary of Inverse Functions

The functions $y = \arcsin x$, $y = \operatorname{arccsc} x$, and $y = \arctan x$ exist in quadrants I and IV in reference to the unit circle.

The functions $y = \arccos x$, $y = arcsec\, x$, and $y = arccot\, x$ exist in quadrants I and II in reference to the unit circle.

STUDY EDGE TIP

These properties apply to $y = \arcsin x$, $y = \operatorname{arccsc} x$, and $y = \arctan x$:

- If you take the inverse of a positive number, it is in quadrant I.
- If you take the inverse of a negative number, it is in quadrant IV.

These properties apply to $y = \arccos x$, $y = \operatorname{arcsec} x$, and $y = \operatorname{arccot} x$:

- If you take the inverse of a positive number, it is in quadrant I.
- If you take the inverse of a negative number, it is in quadrant II.

Remember domain and range restrictions in every case.

1. Evaluate each inverse trigonometric function:

   i. $\sin^{-1}\left(\frac{1}{2}\right)$

   ii. $\sin^{-1}\left(-\frac{1}{2}\right)$

   iii. $\tan^{-1}(\ 1)$

   iv. $\tan^{-1}(-1)$

   v. $\cos^{-1}\left(\frac{1}{2}\right)$

   vi. $\cos^{-1}\left(-\frac{1}{2}\right)$

2. Evaluate each inverse trigonometric function:

   i. $\sin^{-1}\left(\sin\left(\frac{7\pi}{6}\right)\right)$

   ii. $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$

   iii. $\tan^{-1}(\sin\pi)$

3. Evaluate each trigonometric function:

   i. $\sin\left(\arccos\left(-\frac{1}{3}\right)\right)$

   ii. $\cos(\arctan(-2))$