Section 2: Polynomial and Rational Functions

The following maps the videos in this section to the Texas Essential Knowledge and Skills for Mathematics TAC §111.42(c).

2.01 Quadratic Functions
- Precalculus (1)(A)
- Precalculus (1)(B)
- Precalculus (1)(C)
- Precalculus (1)(G)
- Precalculus (2)(F)
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2.02 Complex Numbers
- Precalculus (2)(I)
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2.03 Polynomial and Power Functions
- Precalculus (1)(G)
- Precalculus (2)(F)
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2.04 Long Division
- Precalculus (5)(J)

2.05 Rational Functions
- Precalculus (1)(G)
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- Precalculus (2)(L)
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2.06 Inequalities

- Precalculus (5)(K)
- Precalculus (5)(L)

Note: Unless stated otherwise, any data is fictitious and used solely for the purpose of instruction.

2.01

Quadratic Functions

Quadratic function – A polynomial of degree two whose graph is a parabola

- ***Quadratic form*** – $y = ax^2 + bx + c$

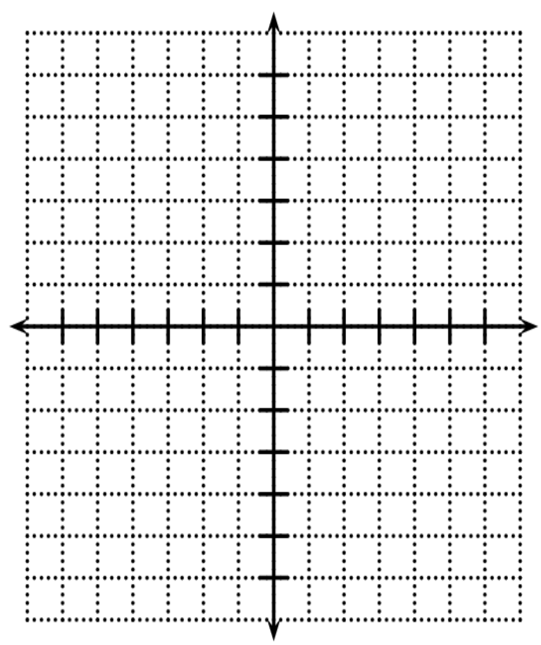
  Vertex is (h, k), where $h = -\frac{b}{2a}$ and $k = f(h)$

- ***Vertex form*** – $y = a(x - h)^2 + k$

  Vertex is (h, k)

- ***Axis of symmetry*** – The x-coordinate of the vertex
- ***Leading coefficient*** – Represented by a
  - If $a > 0$, then the graph opens _______, and the vertex is a ________.
    as $x \to \infty$, $f(x) \to$ and as $x \to -\infty$, $f(x) \to$
  - If $a < 0$, then the graph opens _______, and the vertex is a ________.
    as $x \to \infty$, $f(x) \to$ and as $x \to -\infty$, $f(x) \to$

1. Find the vertex, domain, range, axis of symmetry, and vertex form of $f(x) = x^2 - 4x + 3$.



2. Find the equation of a parabola with a vertex of $(5, -8)$ that passes through the point $(2, 1)$.

STUDY EDGE TIP

When asked to find the maximum or minimum value of a function in a word problem that involves quadratic functions, you are finding the vertex of that equation.

3. Suppose the price and cost of buying Texas Longhorns football jerseys in bulk can be expressed as $p(x) = 400 - \frac{1}{2}x$ and $C(x) = 20x + 300$ respectively. Find the number of jerseys that need to be sold to maximize profit.

4. There are two positive numbers; the sum of the first number and twice the second number is 40. Determine the two numbers that maximize their product.

2.02
Complex Numbers

Imaginary unit – The number represented by i such that $i = \sqrt{-1}$

Complex numbers – The combination of real and imaginary numbers

In the form $a + bi$, a is the real part and bi is the imaginary part.

Notice that powers of the imaginary unit repeat every four terms.

$i^1 =$	$i^2 =$	$i^3 =$	$i^4 =$
$i^5 =$	$i^6 =$	$i^7 =$	$i^8 =$

STUDY EDGE TIP

When the imaginary unit has a large exponent, you can divide the last two digits by 4. The remainder will be equivalent to the new power of i.

1. Find the equivalent expression for each of the following:

   i. i^{137}

   ii. i^{-45}

   iii. i^{2100}

Complex conjugates differ in the sign between the real and imaginary number.

The conjugate of $a + bi$ is $a - bi$. Multiplying them together gives us

2. Find the complex conjugate for each of the following:

   i. $5 - 4i$

   ii. $7 + 3i$

   iii. $3i$

   iv. -7

2.03
Polynomial and Power Functions

Polynomial functions are in the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, where n is a whole number and as are real numbers.

Characteristics of Polynomials

- ***Turn*** – The coordinate where a polynomial has a local maximum or minimum

  A polynomial of degree n has at *most* $n - 1$ turns in its graph.

  Examples:

  second-degree polynomial: third-degree polynomial:

  fourth-degree polynomial: fifth-degree polynomial:

- ***End behavior*** – Where the "arrows" of the graph of the polynomial are pointing as x approaches infinity and negative infinity
  - For even-degree polynomials:
    - If the leading coefficient is positive, the end behavior is that both sides of the graph are up.

    Meaning as $x \to \infty$, $f(x) \to$ and as $x \to -\infty$, $f(x) \to$
    - If the leading coefficient is negative, the end behavior is that both sides of the graph are down.

    Meaning as $x \to \infty$, $f(x) \to$ and as $x \to -\infty$, $f(x) \to$
  - For odd-degree polynomials:
    - If the leading coefficient is positive, the end behavior is down to the left and up to the right.

    Meaning as $x \to \infty$, $f(x) \to$ and as $x \to -\infty$, $f(x) \to$
    - If the leading coefficient is negative, the end behavior is down to the right and up to the left.

    Meaning as $x \to \infty$, $f(x) \to$ and as $x \to -\infty$, $f(x) \to$

- ***Degree of a polynomial***
  - If all like terms of a polynomial are combined, the degree is the highest power.
  - If the polynomial is completely factored, then the degree is found by adding up the multiplicities of the zeroes.
    - ***Zeroes*** – The solution(s) to the equation; also known as the x-intercepts
    - ***Multiplicities*** – The powers on the factors of each zero

1. List the zeroes and multiplicities of each polynomial. Find the degree of each polynomial and its end behavior.

   i. $f(x) = (2x + 3)(x - 2)^3$

   ii. $g(x) = -2x^3(x - 5)(2x - 1)^5$

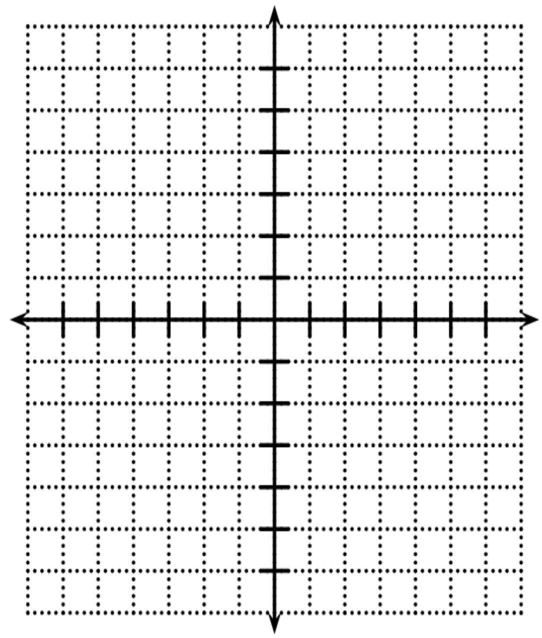
How to Graph a Polynomial Function

1. Find the zeroes, multiplicities of the zeroes, and end behavior.
2. Graph the zeroes, and then graph the end behavior.
3. Use multiplicities to determine if the graph touches or crosses the zero.
   - If the multiplicity is even, it touches the zero.
   - If the multiplicity is odd, it crosses the zero.

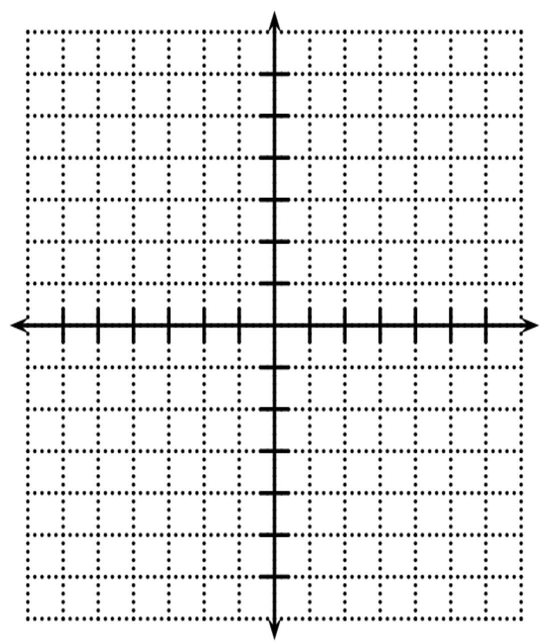
STUDY EDGE TIP

A quick acronym to use when graphing is ZEM: graph based on the **Z**eroes, **E**nd behavior, and **M**ultiplicities.

2. Graph the polynomial function $f(x) = 2(x-1)(x+3)^2$.



3. Graph the polynomial function $f(x) = x^4 - 4x^2$.



2.04

Long Division

Long division is similar to dividing integers, but in the case of polynomial division, you will use 0 as a placeholder for any power missing before the highest power of the polynomial.

1. Use long division to write an equivalent expression for $(9x^3 - 3x^2 + 5) \div (3x^2 + 1)$.

Synthetic division is a shortcut of long division that can be used when the divisor is a binomial in the form $x - k$. Only the coefficients are used for synthetic division.

2. Use synthetic division to write an equivalent expression for $(2x^3 - 4x + 1) \div (x + 2)$.

Remainder Theorem – If you divide a polynomial, $f(x)$, by a factor $x - k$, the remainder is $f(k)$.

Factor Theorem – If $x - k$ is a factor of $f(x)$, then $f(k) = 0$.

3. Which expression is a factor of $f(x) = x^3 - 2x^2 + 1$?

    i. $x - 1$
    ii. $x + 1$
    iii. $x - 2$

Number of Zeroes Theorem – A polynomial of degree n has at most n distinct zeroes.
Example: A polynomial of degree four has at most four different zeroes.

Conjugate Zeroes Theorem – If one of the zeroes of a polynomial is $a + bi$, then $a - bi$ is another zero. For example, if $5 - 4i$ is a zero, then ________ is another zero.

4. Find a polynomial of lowest degree with zeroes of 3, $2i$, 0, and $f(2) = 8$.

2.05

Rational Functions

Rational function – A quotient of two polynomial functions in the form $f(x) = \frac{p(x)}{q(x)}$, where $q(x)$ is not the constant polynomial 0. When compared to polynomial functions, rational functions may have additional traits, like holes and asymptotes.

Hole – A single point where the graph is undefined and indicated visually by an open circle

> To determine holes: When a factor $(x - k)$ is present in both the numerator and denominator, you "cross" these factors out to write an equivalent equation, so the same factor is no longer in the denominator, giving you a hole at $x = k$.

Vertical asymptotes

- $x = a$ will be a vertical asymptote when $f(x) \to \infty$, or $f(x) \to -\infty$ as $x \to a$ from the left side or right side of $x = a$.
- To find the vertical asymptotes, set the denominator equal to zero after you simplify the equation for holes.

Horizontal asymptotes

- $y = b$ will be a horizontal asymptote when $x \to \infty$ or $x \to -\infty$ as $f(x) \to b$.
- To find horizontal asymptotes, compare the degree of the polynomial in the numerator to the degree of the polynomial in the denominator.
  - Case 1: If the degrees are the same, the horizontal asymptote is the ratio of the leading coefficients.
  - Case 2: If the degree of the denominator is larger, the horizontal asymptote will be $y = 0$.
  - Case 3: If the degree of the numerator is larger, there is no horizontal asymptote. Rather, there is an ***oblique asymptote***, which can be found using long division. The equation of the oblique asymptote is the equation before the remainder.

1. Find the holes and asymptotes of each function below.

   i. $f(x) = \frac{x+3}{x^2-9}$

   ii. $g(x) = \frac{2x^2-4x}{x^2-x-2}$

   iii. $h(x) = \frac{x+1}{x^2+1}$

   iv. $j(x) = \frac{x^3+1}{x^2+1}$

STUDY EDGE TIP

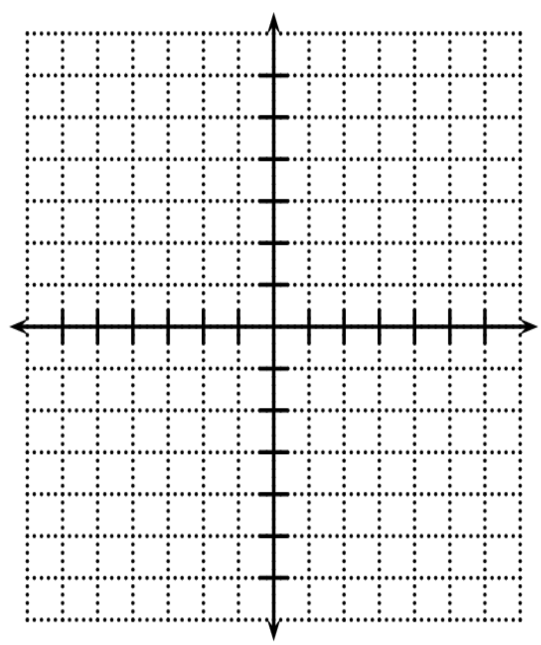
A function can cross or touch its horizontal asymptote.
To find the x value where this occurs, set $f(x) =$ horizontal asymptote, and solve for x.

2. Find where the function intersects its horizontal asymptote. If it does not intersect, then state so.

   i. $f(x) = \frac{x+3}{x^2+9}$

   ii. $g(x) = \frac{x+4}{x-2}$

3. Given the function $f(x) = \frac{3x^2-6x}{x^2+x-6}$, find (a) the domain, (b) the intercepts, (c) the asymptotes, (d) the holes, and (e) where the function crosses the horizontal asymptote. Then, graph the function.



<u>2.06</u>

<u>Inequalities</u>

To solve polynomial or rational inequalities:

Step 1: Move everything to one side, equal the other side to zero, and factor the entire expression.
Step 2: Find the domain and then reduce to the lowest terms.
Step 3: Set the numerator and denominator equal to zero.
Step 4: Place the values from Step 3 and the domain from Step 2 on a number line.
Step 5: Assign open and closed circles.
- In the case of > or <, all are open.
- In the case of ≥ or ≤, all are closed except for numbers not in the domain, which are open.

Step 6: To determine the sign between each number on your number line:
- Plug in a value for x between each interval to determine $+$ or – y value, **<u>or</u>**
- Plug in one number to start, then use multiplicities.
  - Even powers keep the same sign.
  - Odd powers change sign.

Step 7: Write the solution to the inequality in interval notation.

1. Find the interval(s) where the inequality $x^2 - 12 > x$ is true.

2. Find the interval(s) where the inequality $\frac{-2x^5(x-5)^2}{(x+1)(2x-1)^{\frac{2}{3}}} < 0$ is true.

3. Find the domain of the function $f(x) = \sqrt{\frac{(2x-5)^4(3-x)^3}{3x^2(x+1)}}$.

4. Find the interval(s) for which the inequality $\frac{3}{x-5} \leq \frac{4}{x+3}$ is true.