Section 7: Sequence and Series

The following maps the videos in this section to the Texas Essential Knowledge and Skills for Mathematics TAC §111.42(c).

7.01 Sequences and Series Part 1
- Precalculus (1)(E)
- Precalculus (1)(F)

7.02 Sequences and Series Part 2
- Precalculus (1)(E)
- Precalculus (1)(F)
- Precalculus (5)(A)

7.03 Arithmetic Sequences and Series
- Precalculus (1)(E)
- Precalculus (1)(F)
- Precalculus (5)(A)
- Precalculus (5)(B)
- Precalculus (5)(C)
- Precalculus (5)(D)

7.04 Geometric Sequences and Series
- Precalculus (1)(E)
- Precalculus (1)(F)
- Precalculus (5)(A)
- Precalculus (5)(B)
- Precalculus (5)(D)
- Precalculus (5)(E)

7.05 The Binomial Theorem
- Precalculus (5)(F)

Note: Unless stated otherwise, any sample data is fictitious and used solely for the purpose of instruction.

7.01

Sequences and Series Part 1

Integers – The set of numbers with no fractional part, consisting of

- the positive counting numbers $\{1,\ 2,\ 3,\ldots\}$;
- the negative counting numbers $\{-1,\ -2,\ -3,\ldots\}$;
- and zero $\{0\}$.

Sequence – A list of numbers that is written in a specific order so that it has a first member, a second member, a third member, and so on

- The positive counting numbers (also called the positive integers) are an example of a sequence.
- Another example of a sequence is the set of positive odd integers $\{1,\ 3,\ 5,\ 7,\ldots\}$.

Infinite sequence – A function whose domain is the set of positive integers

- The function values $a_1,\ a_2,\ a_3,\ a_4,\ldots,a_n,\ldots$ are the terms of the sequence.
- Sometimes we refer to a finite sequence whose domain consists of only the first n positive integers.
- The terms of a sequence can be listed as $\{a_1,\ a_2,\ a_3,\ a_4,\cdots\}$.
- The terms of a sequence can be described through an explicit formula, such as $a_n = 2n-1$ for the sequence of odd integers. The formula defines the nth term of a sequence as a function of its position in the sequence.
- The variable a_n is called the ***general term*** or ***nth term*** of the sequence.
- When using the subscript notation, n is referred to as the ***index*** of the sequence. The index gives the position of that value in the sequence.

STUDY EDGE TIP

Sometimes it is convenient to begin subscripting a sequence with 0, instead of 1 so that the terms of the sequence start with a_0. When this is the case, the domain includes 0.

1. Write the first three terms of the sequence $a_n = \dfrac{(-1)^n 2^n}{3n-2}$.

2. Write the explicit formula a_n for the sequences below. Assume that n begins at 1.

$\{2, 4, 6, 8, 10, \ldots\}$

$\{4, 2, 1, 0.5, 0.25, \ldots\}$

$\{1, -4, 9, -16, 25, -36, \ldots\}$

$\{1, 2, 6, 24, 120, 720, \ldots\}$

7.02

Sequences and Series Part 2

Recursive sequence – A sequence in which the general term is defined using the preceding term(s) of the sequence

1. Write the first three terms of the recursive sequence defined by $a_n = 2a_{n-1} + 3$ for $n \geq 2$ and $a_1 = 1$.

Factorial – If n is a positive integer, then n factorial is defined as $n! = n(n-1)(n-2)\cdots 2 \cdot 1$ for $n \geq 2$.

- Special cases are $1! = 1$ and $0! = 1$.
- The factorial function is a particular example of a recursive sequence.

2. Simplify the factorial expression $\dfrac{6!}{3!3!}$.

Summation notation for a sum – The sum of the first n terms of a sequence is represented by $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$.

- This is often called summation notation or ***sigma notation***.
- The subscript i is the ***index of summation***. Each consecutive term of the sum is found by increasing the index by one.
- The number 1 is the ***lower limit of summation***. This value determines where the summation begins.
- The subscript n is the ***upper limit of summation***. This value determines where the summation ends.
- The lower limit of summation can be an integer other than 1, and the index of summation can be a letter other than i.

3. Evaluate the following sums:

$$\sum_{n=1}^{3} 2$$

$$\sum_{n=2}^{3} (2n+1)$$

$$\sum_{k=1}^{2} (k^2+1)$$

$$\sum_{n=1}^{4} (-1)^n n^2$$

STUDY EDGE TIP

Variations in the upper and lower limits of summation can produce summation notations that look different, even for the same sum.

Example:

$$\sum_{i=1}^{3} 3^i = 3 + 3^2 + 3^3$$

$$\sum_{i=0}^{2} 3^{i+1} = 3 + 3^2 + 3^3$$

Series – The sum of the terms of a sequence

- The sum of the first n terms is called a finite series, or the ***nth partial sum***, and is denoted by $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$.
- When the values of an infinite sequence are summed, it is called an infinite series and is denoted by $\sum_{i=1}^{\infty} a_i$.

4. Find the third partial sum of the infinite series $\sum_{i=1}^{\infty} 2\left(\frac{1}{3}\right)^i$.

7.03

Arithmetic Sequences and Series

Arithmetic sequence – A sequence in which consecutive terms differ by the same amount

If the sequence is a_1, a_2, a_3, a_4, $\cdots$, a_n, $\cdots$ then there is some number, d, such that $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = d$.

- The value d is called the ***common difference***.
- An example of an arithmetic sequence is $7,\ 12,\ 17,\ 22,\ 27,\ldots$.
- The explicit formula for the nth term of an arithmetic sequence with first term a_1 and common difference d is $a_n = a_1 + (n-1)d$.
- The recursive formula for the nth term of an arithmetic sequence with common difference d is $a_n = a_{n-1} + d$.
- The formula for the common difference is $d = \dfrac{a_n - a_m}{n-m}$ for $n > m$.

1. Consider the sequence $\{5,\ 3,\ 1,-1,-3,...\}$. Find a_{23} and a_n (assuming n starts at 1).

2. Find the eighth term in a sequence with first term $a_1 = 5$ and common difference $d = -2$. Assume n starts at 1.

3. Find the nth term in a sequence with first term $a_1 = 5$ and common difference $d = -2$.

4. Given the arithmetic sequence a_n, find a_1 if $a_{11} = 89$ and $a_{22} = 129$. Determine the recursive formula, a_n, for this sequence.

***n*th partial sum** – The sum of the first n terms of an infinite sequence, denoted by $\sum_{i=1}^{n} a_i$

Arithmetic series – The sum of the first n terms of an arithmetic sequence

- An arithmetic series, therefore, is a partial sum of an arithmetic sequence.
- The sum of the first n terms of an arithmetic sequence with first term a_1 has the formula $S_n = \frac{n}{2}(a_1 + a_n)$. This formula can be used to find a partial sum as well.

5. Find the sum of the finite arithmetic series $3+7+11+15+19+23+27$.

6. Find the partial sum $\sum_{n=1}^{37} 10n$.

7. Use sigma notation to write the sum $4+9+14+19+\cdots+104$. Then evaluate this partial sum.

8. Determine the seating capacity in section 442 of the AT&T Stadium in Arlington, Texas, if there are 30 rows of seats with 20 seats in the first row, 24 seats in the second row, 28 seats in the third row, and so on.

7.04

Geometric Sequences and Series

Geometric sequence – A sequence in which the ratio between consecutive terms is the same. If the sequence is a_1, a_2, a_3, a_4, $\cdots$, a_n, $\cdots$, then there is some number, r, such that

$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \cdots = r$, $r \neq 0$.

- The value r is called the ***common ratio***.
- An example of a geometric sequence is $2,\ 4,\ 8,\ 16,\ 32, \ldots$.
- The explicit formula for the nth term of a geometric sequence with first term a_1 and common ratio r is $a_n = a_1 r^{n-1}$.
- The recursive formula for the nth term of a geometric sequence with first term a_1 and common ratio r is $a_n = r a_{n-1}$, $n \geq 2$.
- The sum of the first n terms of a geometric sequence with first term a_1 and common ratio r is $S_n = a_1 \frac{1 - r^n}{1 - r}$, $r \neq 1$. This is the ***nth partial sum*** of a geometric sequence.

1. Given a geometric sequence with first term $a_1 = 2$ and common ratio $r = -\frac{1}{2}$, find a_5, the fifth term.

2. Given a geometric sequence with first term $a_1 = 3$ and common ratio $r = 2$, find the nth term.

3. If $a_2 = 11$ and $a_3 = 4$ for a given geometric sequence, find the common ratio r.

4. Given the geometric sequence a_n, find a_1 if $a_{11} = 89$ and $a_{12} = 129$. Determine the recursive formula a_n for this sequence.

5. Find the sum of the geometric partial sum $\sum_{n=1}^{7} 3^{n-1}$.

6. Use summation notation to write the sum $2 - \frac{1}{2} + \frac{1}{8} - \cdots + \frac{1}{2048}$.

Geometric series – The summation of all the terms in an infinite geometric sequence

- If the geometric sequence has a common ratio, r, such that $|r| < 1$, then the geometric series $\sum_{n=1}^{\infty} a_1 r^{n-1}$ has the finite sum $\frac{a_1}{1-r}$ and is said to be ***convergent***.
- If the geometric sequence has a common ratio, r, such that $|r| \geq 1$, then the geometric series $\sum_{n=1}^{\infty} a_1 r^{n-1}$ does not have a finite sum and is said to be ***divergent***.

7. Find the sum of the infinite geometric series $\sum_{n=1}^{\infty} 2\left(\frac{1}{2}\right)^{n-1}$ if it exists.

8. Find the sum of the infinite geometric series $\sum_{n=1}^{\infty} 4\left(-\frac{3}{2}\right)^{n-1}$ if it exists.

7.05

The Binomial Theorem

Binomial – A polynomial with two terms

To begin, consider several expansions of $(x+y)^n$ for various values of n:

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

There are several important observations to make about these expansions:

- There are $n+1$ terms in each expression.
- The roles of x and y are symmetric. The powers on x decrease by 1 while the powers on y increase by 1 in successive terms.
- The sum of the exponents on each term is n.
- The coefficients increase and then decrease in a symmetric pattern.

Binomial coefficients – The coefficients in the expansion of a binomial

Binomial theorem – In the expansion of $(x+y)^n$, the coefficient of $x^{n-r}y^r$ is ${}_nC_r = \dfrac{n!}{(n-r)!r!}$.

- The symbol $\begin{pmatrix} n \\ r \end{pmatrix}$ is often used in place of ${}_nC_r$.
- Because of the binomial theorem, it follows that $(x+y)^n = \displaystyle\sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} x^{n-r}y^r$.

1. Evaluate the following:

$_{5}C_{3}$ $\binom{4}{0}$ $\binom{14}{11}$

$_{5}C_{2}$ $\binom{4}{4}$ $\binom{14}{3}$

An important property of the binomial coefficient is its symmetry: $\binom{n}{r} = \binom{n}{n-r}$.

Pascal's triangle

2. Evaluate $\binom{6}{5}$ using Pascal's triangle.

3. Find the expansion of $(x+3y)^4$.

4. Find the coefficient of x^5y^4 in the expansion of $(2x-y)^9$.

5. Determine the fourth term in the expansion of $\left(\sqrt{x}+3\right)^5$.

6. Use the binomial theorem to approximate $(1.01)^4$ to three decimal places.