Section 3: Exponential and Logarithmic Functions

The following maps the videos in this section to the Texas Essential Knowledge and Skills for Mathematics TAC §111.42(c).

3.01 Exponential Functions

- Precalculus (1)(A)
- Precalculus (1)(B)
- Precalculus (1)(C)
- Precalculus (1)(G)
- Precalculus (5)(I)

3.02 Exponential Graphs

- Precalculus (1)(D)
- Precalculus (2)(F)
- Precalculus (2)(G)
- Precalculus (2)(I)
- Precalculus (2)(N)
- Precalculus (2)(J)

3.03 Logarithmic Functions

- Precalculus (1)(B)
- Precalculus (1)(C)
- Precalculus (1)(G)
- Precalculus (5)(G)

3.04 Logarithmic Graphs

- Precalculus (1)(D)
- Precalculus (2)(F)
- Precalculus (2)(G)
- Precalculus (2)(I)
- Precalculus (2)(J)

3.05 Solving Exponential Equations
- Precalculus (1)(A)
- Precalculus (1)(B)
- Precalculus (1)(C)
- Precalculus (1)(D)
- Precalculus (2)(F)
- Precalculus (2)(G)
- Precalculus (2)(J)
- Precalculus (2)(N)
- Precalculus (5)(I)

3.06 Solving Logarithmic Equations
- Precalculus (1)(A)
- Precalculus (1)(B)
- Precalculus (1)(D)
- Precalculus (2)(F)
- Precalculus (2)(G)
- Precalculus (2)(J)
- Precalculus (2)(N)
- Precalculus (5)(G)
- Precalculus (5)(H)

3.07 Compound Interest
- Precalculus (1)(A)
- Precalculus (1)(B)
- Precalculus (1)(C)
- Precalculus (1)(D)
- Precalculus (2)(N)
- Precalculus (5)(I)

3.08 Exponential and Logarithmic Models
- Precalculus (1)(A)
- Precalculus (1)(B)
- Precalculus (1)(C)
- Precalculus (1)(D)
- Precalculus (2)(N)
- Precalculus (5)(H)
- Precalculus (5)(I)

Note: Unless stated otherwise, any sample data is fictitious and used solely for the purpose of instruction.

3.01
Exponential Functions

Exponential Function – A function of the form $f(x) = a^x$, where $a > 0,\ a \neq 1$ and x is any real number

- The value a is called the base of the exponential function.
- We say "a to the x."
- The number $e \approx 2.71$ is an important constant in mathematics, which is often used as the base of an exponential function $f(x) = e^x$.
- e is called the natural base or Euler's number.

1. Evaluate the function $f(x) = 3(2)^{x-2}$ at $x = -1$, $x = 0$, $x = 2$, and $x = \sqrt{7}$.

One-to-One Property – $a^x = a^y$ if and only if $x = y$. This is true only if $a > 0$ and $a \neq 1$.

2. Solve $9 = 3^{x+1}$.

Algebraic Properties of Exponentials

$b^0 = 1$	$b^x b^y = b^{x+y}$	$\dfrac{b^x}{b^y} = b^{x-y}$	$(b^x)^y = b^{xy}$
$(ab)^x = a^x b^x$	$\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$	$b^{-x} = \dfrac{1}{b^x}$	$\dfrac{1}{b^{-x}} = b^x$

Follow the order of operations when working with exponents:
1. Parenthases
2. (Negative) exponents
3. Everything else

3. Simplify $\frac{(2x^2)^3 y^{-4}}{16xy^{-2}}$

Exponential Growth/Decay – A function that models growth/decay by changing at a rate proportional to what is currently present

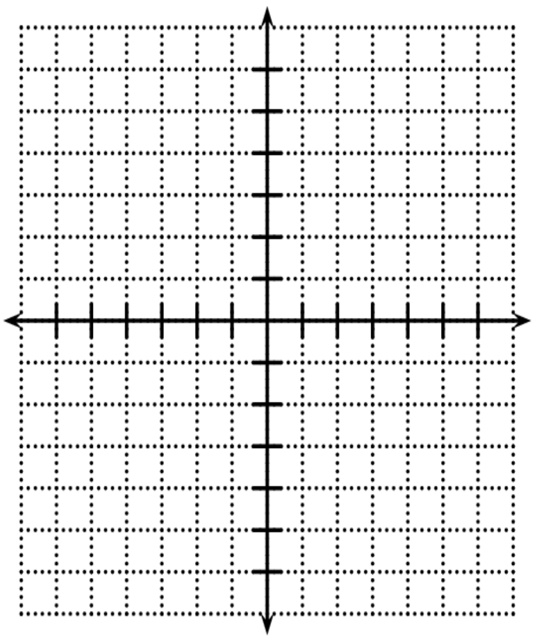
- For any real number x and any positive real numbers a and b such that $b > 1$, an exponential growth function has the form $f(x) = ab^x$.
- For any real number x and any positive real numbers a and b such that $0 < b < 1$, an exponential decay function has the form $f(x) = ab^x$.

4. Population growth in any given region has traditionally followed an exponential growth model, and Texas is no different. According to census data, in 1850, the human population in Texas was 212,592. By 2010, the population had expanded to 25,145,561. Write an algebraic function $N(t)$ representing the population N over time t. Estimate the base of the exponential to three decimal places.
("*Texas Almanac - The Source For All Things Texan Since 1857*," 2011)

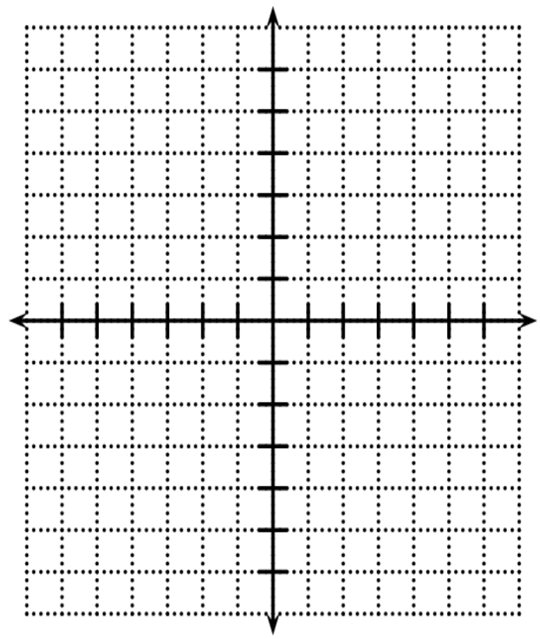
3.02
Exponential Graphs

Consider the function $y = b^x$.

$b > 1$, the function grows



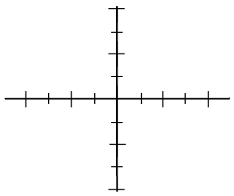
$0 < b < 1$, the function decays



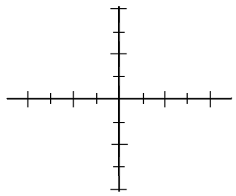
Key Characteristics of $f(x) = 2^x$	**Key Characteristics of $f(x) = \left(\frac{1}{2}\right)^x$**
The domain is $(-\infty, \infty)$	The domain is $(-\infty, \infty)$
The range is $(0, \infty)$	The range is $(0, \infty)$
As $x \to \infty$, $f(x) \to \infty$	As $x \to \infty$, $f(x) \to 0$
As $x \to -\infty$, $f(x) \to 0$	As $x \to -\infty$, $f(x) \to \infty$
$f(x)$ is always increasing	$f(x)$ is always decreasing
$y = 0$ is the horizontal asymptote	$y = 0$ is the horizontal asymptote
y-intercept: $(0,1)$	y-intercept: $(0,1)$

Four Graphs to Know

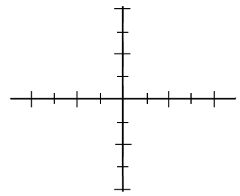
$f(x)=e^{x}$



$f(x)=e^{-x}$



$f(x)=-e^{x}$



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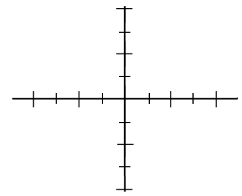


Graph Translations

Given a function $f(x)$, and a number $c>0$:

$f(x)+c$	Shifts $f(x)$ up c units
$f(x)-c$	Shifts $f(x)$ down c units
$f(x+c)$	Shifts $f(x)$ left c units
$f(x-c)$	Shifts $f(x)$ right c units
$f(-x)$	Reflects $f(x)$ over the y-axis
$-f(x)$	Reflects $f(x)$ over the x-axis
$cf(x)$	Stretches $f(x)$ vertically by a factor of $\lvert c\rvert$ if $\lvert c\rvert>1$
$cf(x)$	Compresses $f(x)$ vertically by a factor of $\lvert c\rvert$ if $\lvert c\rvert<1$
$f(cx)$	Stretches $f(x)$ horizontally by a factor of $\lvert 1/c\rvert$ if $\lvert c\rvert<1$
$f(cx)$	Compresses $f(x)$ horizontally by a factor of $\lvert 1/c\rvert$ if $\lvert c\rvert>1$

STUDY EDGE TIP

Overall, there are 3 main characteristics in an exponential graph:

1)

2)

3)

Follow along by navigating to http://tiny.cc/expdesmos in your web browser.

STUDY EDGE TIP

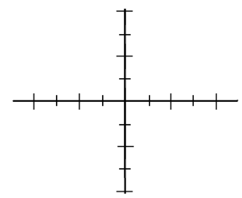
Apply graph translations in the following order:

1) Reflections
2) Stretches and Compressions
3) Shifts

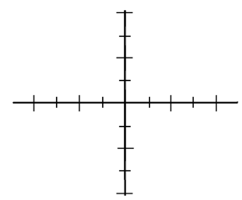
Pick one point on the original graph and apply the translations to it.

1. Graph $f(x) = 2 - 3e^{3-2x}$.

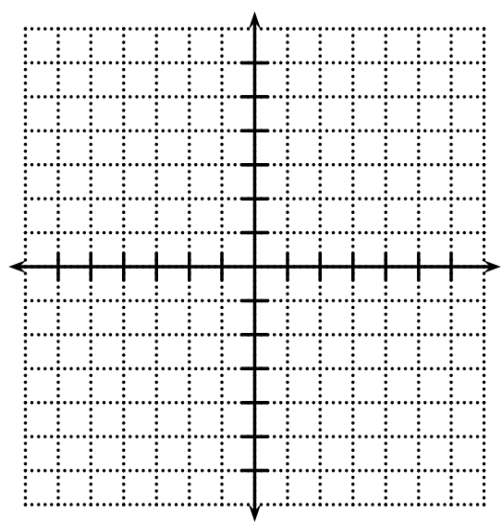
   i. Reflections/Stretches/Compressions:



   ii. Shifts:



   iii. Asymptote(s):



   iv. End Behavior:

   v. Domain:

   vi. Range:

3.03

Logarithmic Functions

Logarithmic Function – A function of the form $f(x) = \log_a(x)$ where $a > 0$, $x > 0$ and $a \neq 1$

- The value a is called the base of the logarithm.
- The inner portion, x, is called the argument of the function. Writing parentheses around the argument is optional, but encouraged.
- We say, "log base a of x".
- The logarithm is the inverse function of the exponential function.
- Logarithms are used as placeholders for exponents that are difficult to manually compute. Mathematically speaking, if $y = \log_a x$, then $a^y = x$.
- <u>You can never take the logarithm of a negative number or zero.</u>

Common Log – If the base of a logarithm is $a = 10$, we call it a common log and use the notation $f(x) = \log(x)$.

Natural Log – If the base of a logarithm is the natural base e, we call it a natural log and use the notation $f(x) = \ln x$.

1. Evaluate each logarithmic expression by converting it to exponential form:

   i. $\log_3 9$

   ii. $\log_{36} 6$

   iii. $\log_{\frac{1}{2}} 8$

For the following formulae, assume that $M, N, b > 0$, $b \neq 1$, and x a real number:

Property	**Logarithm with base b**	**Logarithm with base e**
Product Rule	$\log_b(MN) = \log_b M + \log_b N$	$\ln(MN) = \ln M + \ln N$
Quotient Rule	$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$	$\ln\left(\frac{M}{N}\right) = \ln M - \ln N$
Power Rule	$\log_b\left(M^x\right) = x\log_b M$	$\ln\left(M^x\right) = x\ln M$
Change of Base Formula	$\log_b M = \frac{\log M}{\log b}$	$\log_b M = \frac{\ln M}{\ln b}$
Inverse Properties	$\log_b b^x = x$ $b^{\log_b x} = x,\ x > 0$	$\ln e^x = x$ $e^{\ln x} = x,\ x > 0$
One to One Property	If $\log_b M = \log_b N$, then $M = N$.	If $\ln M = \ln N$, then $M = N$.
General Properties	$\log_b 1 = 0$ $\log_b b = 1$	$\ln 1 = 0$ $\ln e = 1$

2. Find the exact value of each logarithmic expression using properties of logarithms.

i. $5^{-\frac{1}{2}\log_5 16}$

ii. $\frac{3}{\ln\sqrt[4]{e}}$

3. Using properties of logarithms, expand and simplify the logarithmic expression $\ln\left(\frac{x^2}{e^z y^3}\right)$.

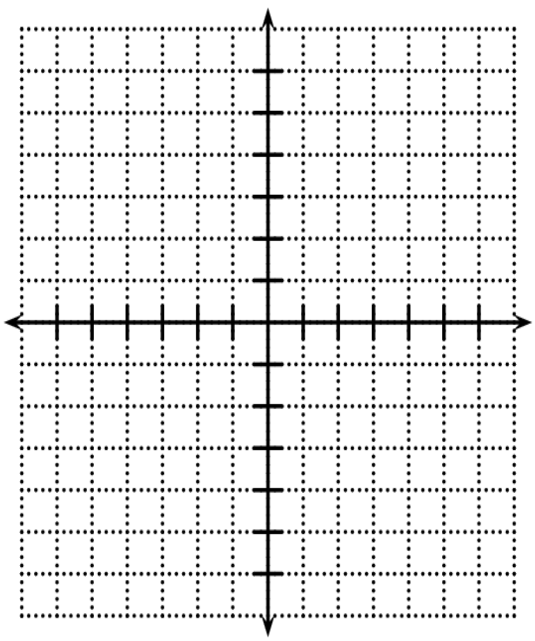
4. Find the inverse function of $f(x) = 3 - \ln(x+2)$.

<u>3.04</u>

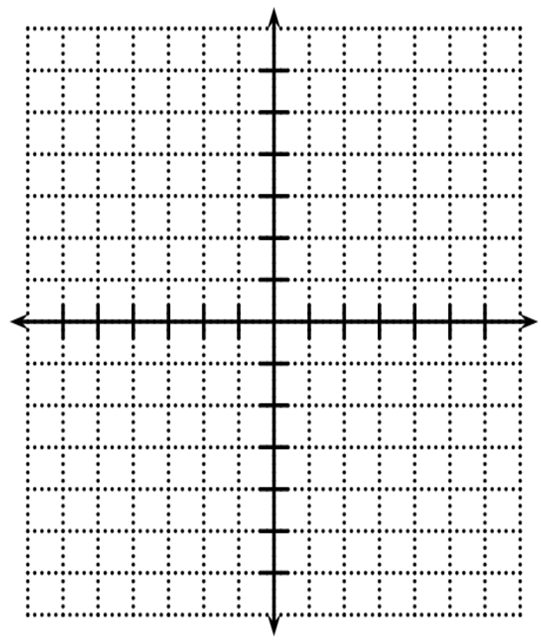
<u>Logarithmic Graphs</u>

Consider the function $y = \log_b x$.

$b > 1$, the function grows



$0 < b < 1$, the function decays



Key Characteristics of $f(x) = \log x$	**Key Characteristics of** $f(x) = \log_{\frac{1}{2}} x$
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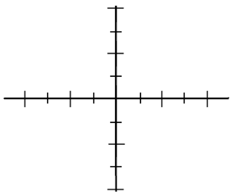
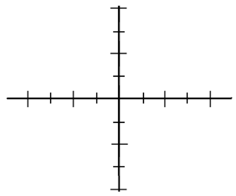
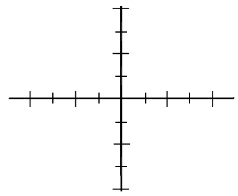
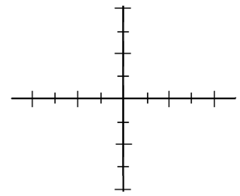
As $x \to 0$, $f(x) \to -\infty$	As $x \to 0$, $f(x) \to \infty$
$f(x)$ is always increasing	$f(x)$ is always decreasing
$x = 0$ is the vertical asymptote	$x = 0$ is the vertical asymptote
x-intercept: $(1,0)$	x-intercept: $(1,0)$

Four Graphs to Know

$f(x)=\ln x$	$f(x)=-\ln x$	$f(x)=\ln(-x)$	$f(x)=-\ln(-x)$
			

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STUDY EDGE TIP

Overall, there are 3 main characteristics in a logarithmic graph:

1)

2)

3)

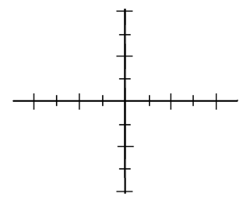
STUDY EDGE TIP

Apply graph translations in the following order:
1) Reflections
2) Stretches and Compressions
3) Shifts

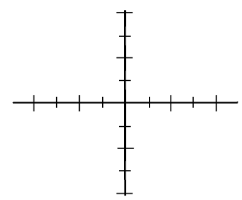
Pick one point on the original graph and apply the translations to it.

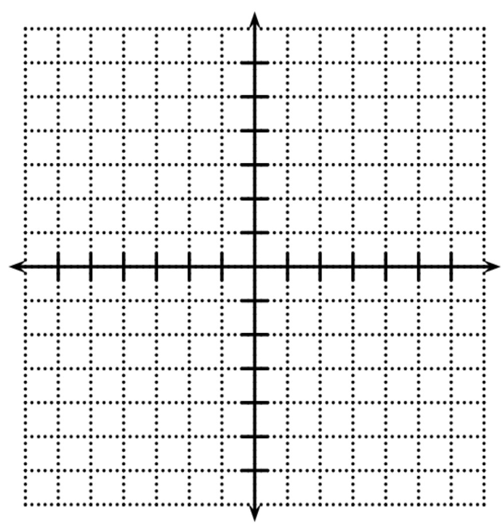
1. Graph $f(x) = -3 - 2\ln\left(\frac{x}{3} + 1\right)$.

   i. Reflections/Stretches/Compressions:



   ii. Shifts:





   iii. Asymptote(s):

   iv. End Behavior:

   v. Domain:

   vi. Range:

3.05

Solving Exponential Equations

Solving Exponential Equations *With* a Common Base

Rewrite each exponential in terms of a common base. Then use the one-to-one property to solve.

One-to-One Property – If $a^x = a^y$, then $x = y$. This is true only if $a > 0$ and $a \neq 1$.

1. Solve $16^{x+4}\left(\frac{1}{2}\right)^{x-5} = \left(2\sqrt{2}\right)^{6x}$ algebraically.

Solving Exponential Equations *Without* a Common Base

Isolate an exponential on each side of the equation. Then, take the natural logarithm of both sides and use properties of logarithms to solve.

2. Solve $e^{3x} = 4^{2x}$ and approximate the solution to three decimal places. Check your solution graphically.

Solving Exponential Equations *With* a Quadratic Form

First, recognize the quadratic form and make a substitution for the linear term. Then, solve the new quadratic and undo the substitution.

3. Solve $e^{2x} - e^{x} - 6 = 0$

4. Several real-world phenomena, such as population growth and financial investments, follow exponential models. An investment's interest compounds continuously according to the equation $A = Pe^{rt}$ where A is the amount earned after investing an initial amount P, r is the interest rate at which the investment grows, and t is the number of years the investment is held. After 10 years of growing an initial investment of $2,000, the account balance is $8,000. What was the interest rate of the investment? Additionally, graph the investment function $A = Pe^{rt}$ and describe its end behavior.

3.06

Solving Logarithmic Equations

When solving equations involving logarithms, verify solutions by substituting them into the original equation. We must ensure that we do ***not*** take the logarithm of zero or a negative number.

Solving Logarithmic Equations Mixed With Constants

Rearrange to isolate all logarithms on one side of the equation. Then, combine the logarithms into a single term using the properties of logs. Set the result equal to the constant value, and then change back to exponential form to solve.

1. Solve the logarithmic equation $\log_8(x+3) = \frac{2}{3} - \log_8(x)$.

Solving Logarithmic Equations *With* a Common Base

Each side of the equation must be turned into a single logarithm using the properties of logs. Then use the inverse property or one-to-one property to solve.

2. Solve $\log(x+2)+\log(x-5)=\log 8$

Solving Logarithmic Equations *Without* a Common Base

Use the change of base formula to find a common base. Then, follow the steps used in solving a logarithmic equation with a common base.

3. Solve $\log_9(x+3)=\log_3(x-3)$

4. Did you know that your hearing is based on logarithms? When twice as many people are talking, you do ***not*** hear the sound twice as loudly! The "loudness" of what you hear follows the model $\beta = 10\log\left(\frac{I}{10^{-12}}\right)$, where your input I is the intensity of sound, and the output β is the "loudness" (measured in decibels, dB). For example, if you were close to an aircraft about to take off, you would experience sound at 110 dB. What is the intensity of sound, I, coming from this aircraft? Graph the logarithmic function $\beta = 10\log\left(\frac{I}{10^{-12}}\right)$ and describe its end behavior.

3.07

Compound Interest

Compounded n **times per year:** $A = P\left(1+\frac{r}{n}\right)^{nt}$
Compounded continuously: $A = Pe^{rt}$
P = Principle (initial amount) A = Final amount t = Number of years r = Interest rate n = Number of times compounded per year Annually; $n = 1$ Semi-annually; $n = 2$ Quarterly; $n = 4$ Monthly; $n = 12$ Daily; $n = 365$

1. Determine the account balance after investing $5,000 over two years, compounded semiannually at an interest rate of 6%.

2. Suppose your goal is to have \$5,000 in eight years by investing in an account that will pay 6% interest compounded monthly. How much money do you need to initially deposit to reach this goal?

3. In how many years will an investment triple that pays out 5% compounded continuously?

3.08

Exponential and Logarithmic Models

Exponential Models

a) Exponential Growth: $y = ae^{bx}$

b) Exponential Decay: $y = ae^{-bx}$

- a is the initial amount when $x = 0$. Note that these equations will sometimes use t instead of x.
- b is the growth/decay rate; $b > 0$.

1. Suppose a population of rabbits doubles every 30 days. If the initial population of rabbits is 250, in how many days will the population reach 2,000?

2. Suppose the half-life of a substance is 400 years. Determine how many years must pass until 20% of the substance remains.

STUDY EDGE TIP

For Exponential Growth/Decay

$$b = \frac{\ln(\text{factor})}{\text{time}} \qquad \text{new time} = \frac{\ln(\text{new factor})}{b}$$

Logistical Growth Model

$y = \dfrac{a}{1 + be^{-rt}}$ for $a, b, r > 0$

3. If a rumor is spread according to the equation $R = \dfrac{1200}{1 + e^{-2t}}$, where t is the elapsed time in years since the rumor originated, and R is the number of people who have heard the rumor, then

    i. How many people initially heard the rumor?

    ii. How long will it take until 1,000 people have heard the rumor?

    iii. What is the maximum number of people who will hear the rumor?

Logarithmic Models

$y = a + b\ln x$, $y = a + b\log x$

4. On the Richter scale, the magnitude R of an earthquake of intensity I is given by $R = \log\frac{I}{I_0}$ where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensity of an earthquake if $R = 4$.

References

Texas Almanac - The Source For All Things Texan Since 1857 (2011, January 11). Retrieved April 11, 2017, from http://texasalmanac.com/topics/population